

GROUP-GRADED ORE EXTENSIONS AND GRADED SIMPLICITY

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ABSTRACT. We study Ore extensions $R[t; \sigma, \delta]$ of rings graded by an arbitrary group Γ . The grading of R extends with t of degree γ if and only if σ is compatible with conjugation by γ and δ with left translation by γ ; the extended grading is then unique. For nonabelian Γ the degree of t need not commute with the degrees of R , and this twist cannot always be removed. If R is gr-simple and σ is an automorphism, then $R[t; \sigma, \delta]$ is gr-simple if and only if it has no homogeneous normal element that is monic of positive t -degree; the quantized Weyl algebra shows that the graded forms of the classical necessary conditions do not suffice. Over a group algebra kG , these extensions correspond to actions of G on the affine line, and gr-simplicity means that the action has no finite orbit. In characteristic zero this is the nonvanishing of a class in $H^1(G, k_\chi)$, and simplicity forces G to be metabelian. We prove the analogous results for Poisson polynomial rings and relate the two settings by semiclassical limits.

1. INTRODUCTION

Ore extensions are noncommutative polynomial rings in one variable [31]. They include skew polynomial rings, rings of differential operators and quantized Weyl algebras. Iterated Ore extensions give many of the standard examples of noncommutative noetherian rings [13, 24]. Two questions about an Ore extension $R[t; \sigma, \delta]$ recur: which structures on R extend to it, and when it is simple. The second question has a long history, from Amitsur's work on differential polynomial rings over simple rings [2] and Jordan's thesis [16] to the work of Cauchon, Lam–Leroy and Öinert–Richter–Silvestrov [6, 21, 30].

In this paper we study both questions for gradings by an arbitrary group. The common thread is the degree of the variable t . Graded Ore extensions are usually $\mathbb{Z}$ -graded, with σ of degree 0 and δ of degree 1 [33]. Many natural rings, however, are graded by nonabelian groups: a group algebra kG is graded by G , and so is every crossed product by G . For such gradings a new phenomenon appears. The defining relation $tr = \sigma(r)t + \delta(r)$ compares tr with $\sigma(r)t$, not with rt . So σ may absorb conjugation by the degree of t , and this degree need not commute with the degrees of R . In the differential case $\sigma = \text{id}$, studied in [1], this cannot happen. We first determine where t can sit (Theorem A). We then describe the graded ideals. Our criterion for gr-simplicity holds for every possible degree of t (Theorem B), and the twist survives in gr-simple rings (Example 2.7). For group algebras the twist can always be removed, and the graded ideals are read off from an action of the group on the affine line (Theorem C).

For example, let a and b be the affine maps $X \mapsto 2X$ and $X \mapsto X + 1$ of $\mathbb{Q}$, and let G be the group they generate, the Baumslag–Solitar group $BS(1, 2)$. The ring generated by $\mathbb{Q}G$ and t with

$$ta = 2at, \quad tb = b^2t + ab$$

is a simple Ore extension of $\mathbb{Q}G$, graded by G with t of the noncentral degree a (Example 3.25). Here σ is not a graded map, since $\sigma(b) = b^2$ has degree $aba^{-1} \neq b$. The ring is simple because G acts faithfully on the affine line with no finite orbit in $\bar{\mathbb{Q}}$ (see the affine actions in Examples 3.22 and 3.25). For the free group on a and b , the analogous ring is only gr-simple, because the free group does not embed in the affine group (Theorem C(4)).

All rings are associative with identity and nonzero, ring homomorphisms preserve the identity, and ideals are two-sided unless stated otherwise. Throughout, k is a field. We write $\text{char } R = 0$ if $n1_R \neq 0$ for every integer $n \geq 1$. Let R be a ring and σ a ring endomorphism of R . A σ -derivation is an additive map $\delta: R \rightarrow R$ such that $\delta(rs) = \delta(r)s + \sigma(r)\delta(s)$ for all $r, s \in R$. The Ore extension $A = R[t; \sigma, \delta]$ is the free left R -module

2020 *Mathematics Subject Classification.* Primary 16S36; Secondary 16W50, 16W25, 16D30, 16S35, 17B63, 20J06.

Key words and phrases. Ore extension, skew derivation, group-graded ring, gr-simple ring, group algebra, group cohomology, skew group ring, Poisson polynomial ring, semiclassical limit.

with basis $1, t, t^2, \dots$, with multiplication determined by

$$tr = \sigma(r)t + \delta(r) \quad (r \in R). \quad (1)$$

We write $R[t; \sigma]$ when $\delta = 0$ and $R[t; \delta]$ when $\sigma = \text{id}$; for the general theory see [13, 24]. Every $f \in A$ can be written uniquely as $f = \sum_i a_i t^i$ with $a_i \in R$. If $f \neq 0$, its t -degree $\deg_t f$ is the largest n with $a_n \neq 0$, and $\text{lc}(f) = a_n$ is its *leading coefficient*. For $n \geq 1$, the element f is *monic of t -degree n* if $f = t^n + \sum_{i < n} a_i t^i$. An element f of a ring S is *normal* if $Sf = fS$; then Sf is an ideal.

For $u \in R$, the map $\delta_u(r) = ur - \sigma(r)u$ is a σ -derivation. A σ -derivation is *inner* if it equals δ_u for some $u \in R$, and *outer* otherwise. If σ is an automorphism, each $\sigma^j \delta \sigma^{-j}$ is again a σ -derivation (Lemma 2.3(3)), and for $n \geq 1$ we put

$$\Delta_n = \sum_{j=0}^{n-1} \sigma^j \delta \sigma^{-j}.$$

This is a σ -derivation, and $\Delta_n = n\delta$ if $\sigma = \text{id}$.

Throughout, Γ is a group with identity e , not necessarily abelian. A Γ -graded ring is a ring $R = \bigoplus_{\tau \in \Gamma} R_\tau$ with $R_\rho R_\tau \subseteq R_{\rho\tau}$ for all ρ, τ ; see [25]. The elements of R_τ are *homogeneous of degree τ* , and $\text{Supp}(R) = \{\tau \in \Gamma : R_\tau \neq 0\}$ is the *support* of R . We write $\deg$ for the Γ -degree and $\deg_t$ for the t -degree. An ideal is *graded* if it is generated by homogeneous elements, and a graded ring is *gr-simple* if 0 and the ring itself are its only graded ideals. For $\gamma \in \Gamma$ put $\iota_\gamma(\tau) = \gamma\tau\gamma^{-1}$. An additive map $f : R \rightarrow R$ is ι_γ -graded if $f(R_\tau) \subseteq R_{\iota_\gamma(\tau)}$ for all $\tau \in \Gamma$; if γ is central, this is the usual notion of a graded map (compare [15, Section 1.1]).

Definition 1.1. Let R be Γ -graded and $\gamma \in \Gamma$. A pair (σ, δ) , where σ is a ring endomorphism of R and δ a σ -derivation, is γ -compatible if

$$\sigma(R_\tau) \subseteq R_{\gamma\tau\gamma^{-1}} \quad \text{and} \quad \delta(R_\tau) \subseteq R_{\gamma\tau} \quad (\tau \in \Gamma). \quad (2)$$

Thus σ is ι_γ -graded, and δ multiplies degrees by γ on the left. Our first result says that compatibility is exactly what is needed in general.

Theorem A (Theorem 2.1). *Let R be Γ -graded and $\gamma \in \Gamma$. The Ore extension $A = R[t; \sigma, \delta]$ admits a Γ -grading extending the grading of R with $t \in A_\gamma$ if and only if (σ, δ) is γ -compatible. Such a grading is unique, and*

$$A_\mu = \bigoplus_{n \geq 0} R_{\mu\gamma^{-n}} t^n \quad (\mu \in \Gamma).$$

For abelian Γ , the conditions (2) say that σ preserves degrees and δ shifts them by γ . For $\Gamma = \mathbb{Z}$ and $\gamma = 1$ they define the usual graded Ore extensions, and for $\sigma = \text{id}$ they recover the criterion of [1, Propositions 2.1 and 2.2], where γ must centralize $\text{Supp}(R)$. For nonabelian Γ , the map σ may move homogeneous components, and γ need not centralize $\text{Supp}(R)$ (Example 2.6). This twist can be removed by a change of variable when R_γ contains a unit (Remark 2.2(3)), but not in general (Examples 2.6 and 2.7). We did not find this extension problem for nonabelian Γ in the literature.

Next we ask when A is gr-simple. An ideal I of R is (σ, δ) -stable if $\sigma(I) \subseteq I$ and $\delta(I) \subseteq I$, and a graded ring R is (σ, δ) -gr-simple if 0 and R are its only (σ, δ) -stable graded ideals. If A is gr-simple, then R is (σ, δ) -gr-simple and δ is outer (Proposition 3.1). These are the graded forms of the standard necessary conditions for simplicity [30, Proposition 4.6 and Remark 4.1]. For $\sigma = \text{id}$ they are also sufficient in characteristic zero. This is due to Jordan for the trivial grading [16], and it holds for arbitrary gradings by [1, Theorem 1.10(2)]. If $\sigma \neq \text{id}$, sufficiency fails even for the trivial grading, since a simple Ore extension of a commutative domain has $\sigma = \text{id}$ [30, Theorem 4.4]. A graded example is the quantized Weyl algebra

$$A = \mathbb{Q}\langle x, t \rangle / (tx - 2xt - 1), \quad \deg x = 1, \quad \deg t = -1.$$

Here A is the Ore extension $\mathbb{Q}[x][t; \sigma, \delta]$, the ring $\mathbb{Q}[x]$ is (σ, δ) -gr-simple and δ is outer, but A is not gr-simple (Proposition 3.2). The obstruction is the homogeneous normal element $1 + xt = tx - xt$, the classical normal element of the quantized Weyl algebra [17]. Such an element can exist because commutators with t need not lower the t -degree: the t^{n+1} -coefficient of $tf - ft$ is $\sigma(a_n) - a_n$ for $f = a_n t^n + \dots$ (Remark 3.3).

Our next result shows that, over a gr-simple coefficient ring, elements of this kind are the only obstruction. We call $f \in A$ a *monic normal element* if f is homogeneous, normal, and monic of positive t -degree.

Theorem B (Theorem 3.7, Corollaries 3.9 and 3.11). *Let R be Γ -graded and gr-simple, let (σ, δ) be γ -compatible with σ an automorphism, and let $A = R[t; \sigma, \delta]$.*

- (1) *A is gr-simple if and only if A contains no monic normal element.*
- (2) *If Δ_n is outer for every $n \geq 1$, then A is gr-simple.*
- (3) *If $\text{char } R = 0$ and $\sigma\delta\sigma^{-1} - \delta$ is inner, then A is gr-simple if and only if δ is outer.*

In the quantized Weyl example, the coefficient ring is only (σ, δ) -gr-simple, and the normal element $1 + xt$ is not monic. The hypothesis that R itself be gr-simple allows the leading coefficient to be normalized in the proof of Theorem B.

Part (3) applies in particular when $\sigma\delta = \delta\sigma$. For the trivial grading these statements are close to known results. Cauchon studied ideals of Ore extensions over quasi-simple rings [6]. For $\sigma = \text{id}$, the argument of part (1) is that of [30, Proposition 4.14]. The argument of part (2) appears in [10, Lemma 4.8]. Part (3) is related to a theorem of Chuang and Tsai on semi-invariant polynomials over prime rings [8] (Remark 3.13); normal and semi-invariant polynomials are also studied in [9, 18, 22]. The graded statements are not covered by these results, because a gr-simple ring need not be prime: the group algebra kC_2 , with $C_2 = \langle g \rangle$, is gr-simple for its natural grading, but $(1 + g)(1 - g) = 0$. We do not know whether the innerness hypothesis in part (3) can be removed (Remark 3.14). In positive characteristic the derivation $\frac{d}{dx}$ of $k[x^{\pm 1}]$ already gives a ring that is not gr-simple, although all the conditions of part (3) except $\text{char } R = 0$ hold (Example 3.12).

Group algebras give a complete picture. Let G be a group. A *character* is a group homomorphism $\chi: G \rightarrow k^\times$, and k_χ denotes k as a G -module via $g \cdot \lambda = \chi(g)\lambda$; see [5] for group cohomology.

Definition 1.2. A *crossed homomorphism* (or *1-cocycle*) for χ is a map $c: G \rightarrow k$ with

$$c(\tau\rho) = c(\tau) + \chi(\tau)c(\rho) \quad (\tau, \rho \in G).$$

The crossed homomorphisms form a k -vector space $Z^1(G, k_\chi)$. The *coboundaries* are the cocycles $x(\chi - 1)$ with $x \in k$; they form a subspace $B^1(G, k_\chi)$, and

$$H^1(G, k_\chi) = Z^1(G, k_\chi) / B^1(G, k_\chi).$$

For $\chi = 1$ this is $H^1(G, k)$ with trivial coefficients, and $Z^1(G, k_1) = \text{Hom}(G, k)$.

The *affine group of the line* is the group $\text{Aff}(1, k) \cong k \rtimes k^\times$ of maps $X \mapsto aX + b$ with $a \in k^\times$ and $b \in k$, under composition. A group is *metabelian* if it has an abelian normal subgroup with abelian quotient; every subgroup of $\text{Aff}(1, k)$ is metabelian. If G acts on a ring S by automorphisms θ_τ , the *skew group ring* $S \rtimes_\theta G = \bigoplus_{\tau \in G} Su_\tau$ has multiplication $(fu_\tau)(gu_\rho) = f\theta_\tau(g)u_{\tau\rho}$, and it is G -graded by $\deg u_\tau = \tau$. Grade kG by G , fix $\gamma \in G$ and a character χ , and put $\sigma(\tau) = \chi(\tau)\gamma\tau\gamma^{-1}$. Every ι_γ -graded k -algebra endomorphism of kG has this form for some χ . The k -linear σ -derivations δ for which (σ, δ) is γ -compatible are exactly the maps $\delta_c(\tau) = c(\tau)\gamma\tau$ with $c \in Z^1(G, k_\chi)$, and δ_c is inner if and only if $c \in B^1(G, k_\chi)$ (Proposition 3.15). The cocycle identity for c says precisely that

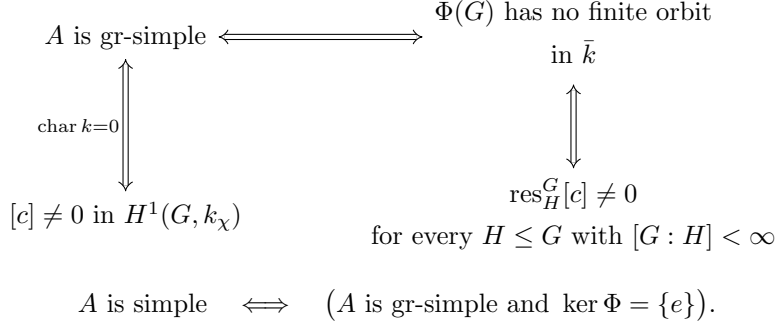
$$\Phi: G \rightarrow \text{Aff}(1, k), \quad \Phi(\tau) = \varphi_\tau, \quad \varphi_\tau(X) = \chi(\tau)X + c(\tau),$$

is a group homomorphism, and every homomorphism $G \rightarrow \text{Aff}(1, k)$ arises in this way.

Theorem C (Proposition 3.20, Theorem 3.21). *Let $A = kG[t; \sigma, \delta_c]$, graded by $\deg t = \gamma$, and let $\bar{k}$ be an algebraic closure of k . Let G act on $k[X]$ by $\theta_\tau(f) = f \circ \varphi_\tau^{-1}$.*

- (1) *The map $f(X)u_\tau \mapsto f(\gamma^{-1}t)\tau$ is an isomorphism of G -graded rings $k[X] \rtimes_\theta G \rightarrow A$.*
- (2) *The graded ideals of A are 0 and the ideals $Af(\gamma^{-1}t)$, where $f \in k[X]$ is monic with $f \circ \varphi_\tau \in k^\times f$ for all $\tau \in G$.*
- (3) *A is gr-simple if and only if $\Phi(G)$ has no finite orbit in $\bar{k}$, if and only if the restriction of $[c]$ to every subgroup of finite index is nonzero. If $\text{char } k = 0$, this holds if and only if $[c] \neq 0$ in $H^1(G, k_\chi)$.*
- (4) *A is simple if and only if it is gr-simple and Φ is injective. In particular, if A is simple, then G is metabelian.*

The following diagram summarizes the simplicity criteria in Theorem C for $A = kG[t; \sigma, \delta_c]$. The unlabelled equivalences hold in every characteristic; the left vertical equivalence requires $\text{char } k = 0$. Simplicity requires, in addition, that the affine action be faithful.



By part (1), the ring A depends on (χ, c) but not on γ . Part (2) is the correspondence between graded ideals of a strongly graded ring and invariant ideals of its identity component [25, Proposition 2.11.7], made explicit, and part (4) rests on a theorem of Öinert on skew group rings [28]; compare [29] for abelian groups. Ore extensions of group algebras also arise as Hopf–Ore extensions [4, 32], where the Hopf structure imposes further conditions; we use no Hopf structure.

In positive characteristic the cohomological criterion fails. If G is finite, then A is never gr-simple, although $H^1(G, k_\chi)$ may be nonzero, for instance for $G = C_p$ in characteristic p (Corollary 3.23). For $G = S_3 \cong \text{Aff}(1, \mathbb{F}_3)$ the obstruction is carried by an orbit rather than by a fixed point (Example 3.22 and Remark 3.24).

Parts (3) and (4) of Theorem C show that gr-simplicity does not imply simplicity. For the free group $F(a, b)$, $\gamma = a$ and $q \in \mathbb{Q} \setminus \{0, 1\}$, the ring with $ta = qat$ and $tb = aba^{-1}t + ab$ is gr-simple but not simple (Examples 3.19 and 3.25), and for $G = \mathbb{Z}^2$ simplicity depends on the arithmetic of the values of c (Example 3.26). In all these examples R_γ contains a unit, so the degree of t can be moved to e by the change of variable $s = \gamma^{-1}t$. The twist can also coexist with gr-simplicity. In Example 2.7, Γ is the fundamental group of the Klein bottle, σ is not graded, $R_\gamma = 0$, and A is gr-simple but not simple. Example 2.6 also has an intrinsic twist, but it is not gr-simple.

The same questions make sense for Poisson algebras. Let P be a Poisson algebra over k , let α be a Poisson derivation of P and δ a Poisson α -derivation. By a theorem of Oh [26], in the sign convention of [23], $P[t]$ has a unique Poisson bracket that extends the bracket of P and satisfies $\{t, a\} = \alpha(a)t + \delta(a)$; this is the *Poisson polynomial ring* $B = P[t; \alpha, \delta]_P$. The derivation δ is α -inner if $\delta(a) = \{u, a\} - u\alpha(a)$ for some $u \in P$, and an element $f \in B$ is *Poisson normal* if $\{B, f\} \subseteq fB$. In a commutative graded ring, nonzero products and brackets involve only commuting degrees, so the twist ι_γ of Theorem A disappears, and we take Γ abelian. The twist also disappears in semiclassical limits (Remark 4.7). A pair (α, δ) is γ -compatible if $\alpha(P_\tau) \subseteq P_\tau$ and $\delta(P_\tau) \subseteq P_{\gamma\tau}$ for all τ , and P is *Poisson gr-simple* if 0 and P are its only graded Poisson ideals. The precise definitions are recalled in Section 4.

Theorem D (Proposition 4.1, Theorem 4.3). *Let Γ be abelian, P a Γ -graded Poisson algebra, $\gamma \in \Gamma$ and $B = P[t; \alpha, \delta]_P$.*

- (1) *B admits a grading extending that of P with $t \in B_\gamma$ if and only if (α, δ) is γ -compatible.*
- (2) *Assume that this holds and that P is Poisson gr-simple. Then B is Poisson gr-simple if and only if B contains no homogeneous Poisson normal element $t^n + \sum_{i < n} b_i t^i$ with $n \geq 1$.*
- (3) *If moreover $\text{char } k = 0$, then B is Poisson gr-simple if and only if δ is not α -inner.*

Part (3) needs no analogue of the innerness hypothesis of Theorem B(3). The reason is that the t^{n-1} -coefficient of $\{a, t^n\}$ is $-n\delta(a)$, whereas in the skew case the corresponding coefficient involves Δ_n (Lemma 3.6(2)). Simplicity criteria for Poisson Ore extensions based on Poisson normal elements also appear in [27]. Semiclassical limits connect the two settings. The quantized Weyl algebra above lies in the graded family $zx = qxz + (q - 1)$, whose semiclassical limit is the Poisson polynomial ring with $\{z, x\} = xz + 1$. The element $1 + xz$ is normal in the family and Poisson normal in the limit (Example 4.6). Moreover, if $\delta = \delta_u$ is inner, then its semiclassical limit is α -inner (Lemma 4.5).

The paper is organized as follows. In Section 2 we prove Theorem A, study γ -compatible pairs and construct the nonabelian examples. Section 3 is devoted to gr-simplicity and proves Theorems B and C. Section 4 treats Poisson polynomial rings and semiclassical limits, and proves Theorem D.

2. GRADINGS ON ORE EXTENSIONS

Throughout this section, R is a Γ -graded ring, σ is a ring endomorphism of R , δ is a σ -derivation, and $A = R[t; \sigma, \delta]$.

Theorem 2.1. *Let $\gamma \in \Gamma$. The following are equivalent:*

- (i) *A admits a Γ -grading $\{A_\mu\}$ with $R_\tau \subseteq A_\tau$ for all τ and $t \in A_\gamma$.*
- (ii) *(σ, δ) is γ -compatible.*

In that case the grading in (i) is unique, namely

$$A_\mu = \bigoplus_{n \geq 0} R_{\mu\gamma^{-n}} t^n \quad (\mu \in \Gamma). \quad (3)$$

Moreover, if $\delta \neq 0$, then γ is determined by δ .

Proof. (ii) $\Rightarrow$ (i). Define A_μ by (3). Since $A = \bigoplus_n R t^n$ is free over R and $R = \bigoplus_\rho R_\rho$, we have $A = \bigoplus_{n,\rho} R_\rho t^n$. Placing $R_\rho t^n$ in $A_{\rho\gamma^n}$ gives $A = \bigoplus_\mu A_\mu$, and $1 \in A_e$. It remains to prove $A_\mu A_\beta \subseteq A_{\mu\beta}$.

First, $R_\rho A_\beta \subseteq A_{\rho\beta}$, because $R_\rho R_{\beta\gamma^{-n}} \subseteq R_{\rho\beta\gamma^{-n}}$. Next, $tA_\beta \subseteq A_{\gamma\beta}$: for $r \in R_\rho$ and $n \geq 0$ we have $rt^n \in A_{\rho\gamma^n}$, and by (1) and (2),

$$trt^n = \sigma(r)t^{n+1} + \delta(r)t^n \in R_{\gamma\rho\gamma^{-1}}t^{n+1} + R_{\gamma\rho}t^n \subseteq A_{\gamma\rho\gamma^n}.$$

By induction on m , $t^m A_\beta \subseteq A_{\gamma^m\beta}$ for all $m \geq 0$: if it holds for m , then $t^{m+1}A_\beta \subseteq tA_{\gamma^m\beta} \subseteq A_{\gamma^{m+1}\beta}$. Hence $(rt^m)A_\beta \subseteq R_\rho A_{\gamma^m\beta} \subseteq A_{\rho\gamma^m\beta}$ for $r \in R_\rho$. Since A_μ is spanned by such rt^m with $\rho\gamma^m = \mu$, the grading is multiplicative.

(i) $\Rightarrow$ (ii). Let $0 \neq r \in R_\tau$, and write $\sigma(r) = \sum_\rho s_\rho$ and $\delta(r) = \sum_\rho d_\rho$ with $s_\rho, d_\rho \in R_\rho$. By (i), $s_\rho t \in A_{\rho\gamma}$, $d_\rho \in A_\rho$ and $tr \in A_{\gamma\tau}$. For $\beta \neq \gamma\tau$, the component of $tr = \sum_\rho s_\rho t + \sum_\rho d_\rho$ of degree β is $s_{\beta\gamma^{-1}}t + d_\beta$, and it vanishes. Since $1, t$ are R -linearly independent, $s_{\beta\gamma^{-1}} = 0 = d_\beta$. Hence $\sigma(r) \in R_{\gamma\tau\gamma^{-1}}$ and $\delta(r) \in R_{\gamma\tau}$.

Uniqueness. Let $\{A'_\mu\}$ be a grading as in (i). Then $t^n \in A'_{\gamma^n}$, so $R_\rho t^n \subseteq A'_{\rho\gamma^n}$ and $A_\mu \subseteq A'_\mu$ for every μ . If $a \in A'_\mu$, write $a = \sum_\beta a_\beta$ with $a_\beta \in A_\beta \subseteq A'_\beta$; uniqueness of homogeneous components in $\{A'_\beta\}$ gives $a = a_\mu \in A_\mu$. Finally, if $\delta \neq 0$, choose a homogeneous $x \in R_\tau$ with $\delta(x) \neq 0$. If γ and γ' both satisfy (2), then $0 \neq \delta(x) \in R_{\gamma\tau} \cap R_{\gamma'\tau}$, so $\gamma = \gamma'$. $\square$

Remark 2.2. (1) For nonabelian Γ , σ is ι_γ -graded but need not be graded: $\sigma(R_\tau) \subseteq R_\tau$ for all τ if and only if $\gamma\tau\gamma^{-1} = \tau$ whenever $\sigma(R_\tau) \neq 0$. In Examples 2.6, 2.7 and 3.19, σ is not graded.

(2) If $\sigma = \text{id}$, then $R_\tau \subseteq R_{\gamma\tau\gamma^{-1}}$ forces $\gamma\tau\gamma^{-1} = \tau$ for $\tau \in \text{Supp}(R)$. So γ centralizes $\text{Supp}(R)$ and $\delta(R_\tau) \subseteq R_{\gamma\tau} = R_{\tau\gamma}$. This recovers [1, Proposition 2.1].

(3) Let (σ, δ) be γ -compatible and let $g \in R_\gamma$ be a unit. Then $g^{-1} \in R_{\gamma^{-1}}$ (compare components of $gg^{-1} = 1$). Put $t' = g^{-1}t \in A_e$, $\sigma'(r) = g^{-1}\sigma(r)g$ and $\delta' = g^{-1}\delta$. Then $t'r = \sigma'(r)t' + \delta'(r)$, σ' is a ring endomorphism, and

$$\delta'(rr') = g^{-1}(\sigma(r)\delta(r') + \delta(r)r') = \sigma'(r)\delta'(r') + \delta'(r)r',$$

so δ' is a σ' -derivation. Moreover $\sigma'(R_\tau) \subseteq R_{\gamma^{-1}}R_{\gamma\tau\gamma^{-1}}R_\gamma \subseteq R_\tau$ and $\delta'(R_\tau) \subseteq R_{\gamma^{-1}}R_{\gamma\tau} \subseteq R_\tau$. Since $t'^{n+1} = g^{-1}t t^n$, induction gives $t'^n = u_n t^n + (\text{lower terms})$ with $u_{n+1} = g^{-1}\sigma(u_n)$, that is,

$$u_n = g^{-1}\sigma(g^{-1}) \cdots \sigma^{n-1}(g^{-1}),$$

a unit. So $\{t'^n\}_{n \geq 0}$ is a left R -basis of A , and $A = R[t'; \sigma', \delta']$ with $\deg t' = e$ and σ' graded. In particular, if $\sigma(r) = grg^{-1}$ for all r , then $\sigma' = \text{id}$ and A is the graded differential polynomial ring $R[t'; \delta']$. Thus, if the degree of t cannot be moved to e in this way, then R_γ contains no unit (Examples 2.6 and 2.7).

The change of variable in Remark 2.2(3) gives the following commutative diagram of graded rings. The isomorphism fixes R , and the slanted arrows are the natural inclusions. A homogeneous unit $g \in R_\gamma$ carries the degree of t , so $t' = g^{-1}t$ has degree e . The two presentations describe the same graded extension of R . This normalization requires such a unit; Examples 2.6 and 2.7 show that it need not be possible in general.

$$\begin{array}{ccc}
R[t; \sigma, \delta] & \xrightarrow[\substack{\sim \\ t \mapsto gt'}]{} & R[t'; \sigma', \delta'] \\
& \nwarrow \quad \nearrow & \\
& R &
\end{array}$$

$$t' = g^{-1}t, \quad \deg t = \gamma, \quad \deg t' = e,$$

$$\sigma'(r) = g^{-1}\sigma(r)g, \quad \delta'(r) = g^{-1}\delta(r), \quad \sigma'(R_\tau), \delta'(R_\tau) \subseteq R_\tau.$$

Lemma 2.3. *Let (σ, δ) be γ -compatible.*

- (1) *For $u \in R_\gamma$, the pair (σ, δ_u) is γ -compatible.*
- (2) *If $\delta = \delta_u$ with $u = \sum_\rho u_\rho$ and $u_\rho \in R_\rho$, then $\delta = \delta_{u_\gamma}$. In particular, δ is inner if and only if $\delta = \delta_u$ for some $u \in R_\gamma$.*
- (3) *If σ is an automorphism, then $\sigma^j(R_\tau) = R_{\gamma^j\tau\gamma^{-j}}$ for all $j \in \mathbb{Z}$ and $\tau \in \Gamma$. Each $\sigma^j\delta\sigma^{-j}$ is a σ -derivation, and the pair $(\sigma, \sigma^j\delta\sigma^{-j})$ is γ -compatible for every $j \in \mathbb{Z}$. Moreover, $\sigma\delta_u\sigma^{-1} = \delta_{\sigma(u)}$, and $\sigma(u) \in R_\gamma$ for $u \in R_\gamma$.*

Proof. (1) For $r \in R_\tau$ we have $ur \in R_{\gamma\tau}$ and $\sigma(r)u \in R_{\gamma\tau\gamma^{-1}}R_\gamma \subseteq R_{\gamma\tau}$, so $\delta_u(r) \in R_{\gamma\tau}$.

(2) Let $r \in R_\tau$. Then $u_\rho r \in R_{\rho\tau}$ and $\sigma(r)u_\rho \in R_{\gamma\tau\gamma^{-1}\rho}$. Each of these degrees equals $\gamma\tau$ only for $\rho = \gamma$. Since $\delta(r) \in R_{\gamma\tau}$, comparing components of degree $\gamma\tau$ in $\delta(r) = \sum_\rho (u_\rho r - \sigma(r)u_\rho)$ gives $\delta(r) = u_\gamma r - \sigma(r)u_\gamma$. By additivity, $\delta = \delta_{u_\gamma}$.

(3) We first show $\sigma(R_\tau) = R_{\gamma\tau\gamma^{-1}}$. The inclusion $\subseteq$ is compatibility. Let $x \in R_{\gamma\tau\gamma^{-1}}$ and write $x = \sigma(y)$ with $y = \sum_\rho y_\rho$. The terms $\sigma(y_\rho) \in R_{\gamma\rho\gamma^{-1}}$ lie in pairwise distinct degrees, so $x = \sigma(y_\tau)$. Hence $\sigma(R_\tau) = R_{\gamma\tau\gamma^{-1}}$ and, σ being bijective, $\sigma^{-1}(R_\rho) = R_{\gamma^{-1}\rho\gamma}$. Induction on $|j|$ gives $\sigma^j(R_\tau) = R_{\gamma^j\tau\gamma^{-j}}$ for all $j \in \mathbb{Z}$.

Put $\delta_j = \sigma^j\delta\sigma^{-j}$. Since $\sigma^{\pm j}$ are ring homomorphisms,

$$\delta_j(ab) = \sigma^j(\sigma^{1-j}(a)\delta(\sigma^{-j}(b)) + \delta(\sigma^{-j}(a))\sigma^{-j}(b)) = \sigma(a)\delta_j(b) + \delta_j(a)b,$$

so δ_j is a σ -derivation. For $a \in R_\tau$ we get $\sigma^{-j}(a) \in R_{\gamma^{-j}\tau\gamma^j}$, then $\delta\sigma^{-j}(a) \in R_{\gamma^{1-j}\tau\gamma^j}$, and then $\delta_j(a) \in R_{\gamma^j\gamma^{1-j}\tau\gamma^j\gamma^{-j}} = R_{\gamma\tau}$. Finally, applying σ to $\delta_u(\sigma^{-1}(r)) = u\sigma^{-1}(r) - r\sigma^{-1}(r)$ gives $\sigma\delta_u\sigma^{-1}(r) = \sigma(u)r - \sigma(r)\sigma(u) = \delta_{\sigma(u)}(r)$, and $\sigma(u) \in R_{\gamma\gamma\gamma^{-1}} = R_\gamma$. $\square$

As in the ungraded case [11, Lemma 1.5], the inner case reduces to a skew polynomial ring.

Lemma 2.4. *Let (σ, δ) be γ -compatible with $\delta = \delta_u$, $u \in R_\gamma$. Then $s = t - u \in A_\gamma$ satisfies $sr = \sigma(r)s$ for $r \in R$, $A = \bigoplus_{n \geq 0} Rs^n$ is the graded skew polynomial ring $R[s; \sigma]$ with $\deg s = \gamma$, and As is a nonzero proper graded ideal of $\bar{A}$.*

Proof. Since $\delta(r) = ur - \sigma(r)u$,

$$sr = \sigma(r)t + \delta(r) - ur = \sigma(r)t - \sigma(r)u = \sigma(r)s.$$

By (1), $t^i u \in \sigma^i(u)t^i + \sum_{m < i} Rt^m$, and induction on n gives $s^n - t^n \in \sum_{m < n} Rt^m$. Hence $\{s^n\}_{n \geq 0}$ is a left R -basis of A . Since $s(rs^n) = \sigma(r)s^{n+1} \in As$, we get $sA \subseteq As$, so the left ideal As is two-sided. In the basis $\{s^n\}$ it equals $\bigoplus_{n \geq 1} Rs^n$, so $1 \notin As$. It is graded because s is homogeneous. $\square$

Example 2.5 (Quantized Weyl algebra). Let $R = \mathbb{Q}[x]$ with $\deg x = 1$, $\sigma(f)(x) = f(2x)$ and $\delta(f) = (f(2x) - f(x))/x$. Then $\delta(fg) = \sigma(f)\delta(g) + \delta(f)g$, σ preserves degrees, and

$$\delta(x^m) = (2^m - 1)x^{m-1} \quad (m \geq 1).$$

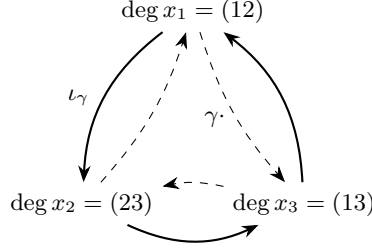
Thus (σ, δ) is (-1) -compatible. Since $tx = 2xt + 1$, we have

$$A = \mathbb{Q}[x][t; \sigma, \delta] \cong \mathbb{Q}\langle x, t \rangle / (tx - 2xt - 1),$$

and by Theorem 2.1 the ring A is graded by $\deg x = 1$ and $\deg t = -1$.

In the next example σ moves homogeneous components and t has noncentral degree.

In the diagram below, the vertices are the degrees of x_1, x_2, x_3 in Example 2.6, with $\gamma = (123)$. Solid arrows represent $\tau \mapsto \gamma\tau\gamma^{-1}$, the effect of σ on degrees, whereas dashed arrows represent $\tau \mapsto \gamma\tau$, the effect of δ . These maps rotate the three transpositions in opposite directions, and γ commutes with none of them. Permutations are composed from right to left.



Example 2.6 (A free algebra graded by S_3). Compose permutations from right to left. Let $\Gamma = S_3$, $\gamma = (123)$, and let $R = k\langle x_1, x_2, x_3 \rangle$ be the free algebra graded by

$$\deg x_1 = (12), \quad \deg x_2 = (23), \quad \deg x_3 = (13).$$

Then $\gamma(12)\gamma^{-1} = (23)$, $\gamma(23)\gamma^{-1} = (13)$, $\gamma(13)\gamma^{-1} = (12)$, and $\gamma(12) = (13)$, $\gamma(23) = (12)$, $\gamma(13) = (23)$. Let σ be the automorphism $x_1 \mapsto x_2 \mapsto x_3 \mapsto x_1$, and let δ be the σ -derivation with

$$\delta(x_1) = x_3, \quad \delta(x_2) = x_1, \quad \delta(x_3) = x_2;$$

it exists and is unique because R is free. Thus $\deg \sigma(x_i) = \gamma \deg(x_i) \gamma^{-1}$ and $\deg \delta(x_i) = \gamma \deg(x_i)$. Both properties pass to products: if $u \in R_\mu$ and $v \in R_\beta$ satisfy them, then $\sigma(uv) \in R_{\gamma\mu\beta\gamma^{-1}}$ and $\delta(uv) = \sigma(u)\delta(v) + \delta(u)v \in R_{\gamma\mu\beta}$. Hence (σ, δ) is γ -compatible, and $A = R[t; \sigma, \delta]$ is graded with $\deg t = (123)$.

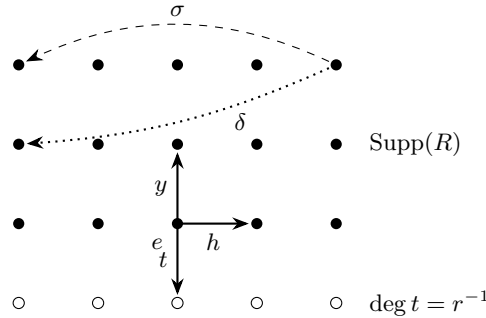
The transpositions generate S_3 , whose centre is trivial, so no $\gamma \neq e$ centralizes $\text{Supp}(R)$. By Remark 2.2(2), a differential polynomial ring over R admits a compatible grading only with $\deg t = e$. The homogeneous units of R are the nonzero scalars, so R_γ contains no unit and Remark 2.2(3) does not apply.

The derivation δ is outer. The degree of a monomial of length ℓ is a permutation of sign $(-1)^\ell$, and γ is even, so every monomial in R_γ has even length $\ell \geq 2$. For $u \in R_\gamma$, every monomial of $\delta_u(x_1) = ux_1 - x_2u$ therefore has length at least 3, whereas $\delta(x_1) = x_3$. Hence $\delta \neq \delta_u$ for $u \in R_\gamma$, so δ is outer by Lemma 2.3(2). On the other hand, the ideal generated by x_1, x_2, x_3 is graded and stable under σ and δ , so A is not gr-simple by Proposition 3.1 below.

The degree of t cannot be moved to e : no $s \in A_e$ has powers forming a left R -basis of A . Suppose that $s \in A_e$ is such an element. Since R is a domain and σ is injective, $\text{lc}(fg) = \text{lc}(f)\sigma^{\deg_t f}(\text{lc}(g)) \neq 0$, so $\deg_t(fg) = \deg_t f + \deg_t g$ for nonzero $f, g \in A$. Then $d = \deg_t s \geq 1$ because $A \neq R$, and every nonzero element of $\bigoplus_n Rs^n$ has t -degree divisible by d ; applied to t , this gives $d = 1$. Write $s = bt + a$ and $t = c_1s + c_0$ with $a, b, c_0, c_1 \in R$. By (3), $b \in R_{\gamma^{-1}}$, and comparing t -coefficients gives $c_1b = 1$. As R is a domain, $(bc_1 - 1)b = 0$ gives $bc_1 = 1$, so b is a unit of the free algebra, hence a nonzero scalar, and $b \in R_e$. This contradicts $b \in R_{\gamma^{-1}}$ with $\gamma^{-1} \neq e$.

The next example shows that the twist can be intrinsic in a gr-simple extension.

The diagram below displays a finite window of the grading in Example 2.7, placing $h^m r^n$ at (m, n) . Filled dots represent degrees in $\text{Supp}(R) = \{h^m r^n : m \in \mathbb{Z}, n \geq 0\}$; hollow dots represent the excluded row $n = -1$. The arrows from e give the degrees of h , y and t . In particular, $\deg t = \gamma = r^{-1} \notin \text{Supp}(R)$, so $R_\gamma = 0$. On degrees, σ sends (m, n) to $(-m, n)$. For $n \geq 1$, δ sends degree (m, n) to $(-m, n-1)$, and it vanishes on the row $n = 0$.



Example 2.7 (A gr-simple extension with an intrinsic twist). Let $\Gamma = \langle h, r \mid rhr^{-1} = h^{-1} \rangle$ be the fundamental group of the Klein bottle; every element of Γ is uniquely $h^m r^n$ with $m, n \in \mathbb{Z}$. Let $R = \mathbb{Q}[h^{\pm 1}][y; \theta]$ with

$\theta(h) = h^{-1}$, graded by $\deg h = h$ and $\deg y = r$. Thus $\{h^m y^i\}$ is a $\mathbb{Q}$ -basis of R , $h^m y^i \in R_{h^m r^i}$, and

$$(h^m y^i)(h^n y^j) = h^{m+(-1)^i n} y^{i+j}.$$

Put $\gamma = r^{-1}$ and define $\mathbb{Q}$ -linear maps by

$$\sigma(h^m y^i) = h^{-m} y^i, \quad \delta(h^m) = 0, \quad \delta(h^m y^i) = i h^{-m} y^{i-1} \quad (i \geq 1).$$

Both sides of $\sigma(ab) = \sigma(a)\sigma(b)$ equal $h^{-m-(-1)^i n} y^{i+j}$ for $a = h^m y^i$ and $b = h^n y^j$, so σ is an automorphism with $\sigma^2 = \text{id}$. Since $(-1)^{i-1} = -(-1)^i$,

$$\sigma(a)\delta(b) + \delta(a)b = j h^{-m-(-1)^i n} y^{i+j-1} + i h^{-m+(-1)^{i-1} n} y^{i+j-1} = \delta(ab),$$

so δ is a σ -derivation. As $\gamma h^m r^i \gamma^{-1} = h^{-m} r^i$ and $\gamma h^m r^i = h^{-m} r^{i-1}$, the pair (σ, δ) is γ -compatible, and $A = R[t; \sigma, \delta]$ is graded with $\deg t = r^{-1}$. Its defining relations are

$$yh = h^{-1}y, \quad th = h^{-1}t, \quad ty = yt + 1.$$

Here γ does not centralize $h \in \text{Supp}(R)$, so σ is not graded, and $R_\gamma = 0$.

The ring A is gr-simple. Let W be the subalgebra generated by y and t . The elements $h^m y^i t^j$ form a $\mathbb{Q}$ -basis of A , so $W \cong A_1(\mathbb{Q})$, and $h^m y^i t^j \in A_{h^m r^i - j}$ gives $A_{h^m r^n} = h^m W_n$ with $W_n = \text{span}\{y^i t^j : i - j = n\}$. Let $J \neq 0$ be a graded ideal and $0 \neq f \in J_{h^m r^n}$. Then $h^{-m} f \in J \cap W$, so $J \cap W$ is a nonzero ideal of the simple ring $A_1(\mathbb{Q})$. Hence $1 \in J$.

The ring A is not simple. The element $h + h^{-1}$ commutes with h , y and t , so it is central. Since R is a domain and σ is injective, the t -degree is additive on A (see Example 2.6), so the units of A lie in R ; likewise the units of R lie in $\mathbb{Q}[h^{\pm 1}]$ and are the elements λh^m with $\lambda \in \mathbb{Q}^\times$. Hence $h + h^{-1} - 2$ is a central nonunit, and $A(h + h^{-1} - 2)$ is a nonzero proper ideal.

The degree of t cannot be moved to e . By (3), $A_e = \bigoplus_n \mathbb{Q} y^n t^n$. The first part of the argument of Example 2.6 shows that an element $s \in A_e$ whose powers form a left R -basis has t -degree one. So $s = b y t + a$ with $a, b \in \mathbb{Q}$ and $b \neq 0$, and $t = c_1 s + c_0$ with $c_0, c_1 \in R$ forces $c_1 b y = 1$ in R , which is impossible since y is not a unit.

Note that R is not gr-simple, since $Ry = yR$ is a proper graded ideal, so Theorem 3.7 does not apply.

3. GR-SIMPLICITY

Throughout this section (σ, δ) is γ -compatible and A carries the grading (3).

3.1. Necessary conditions and a counterexample.

Proposition 3.1. *If A is gr-simple, then R is (σ, δ) -gr-simple and δ is outer.*

Proof. Let I be a graded ideal of R with $\sigma(I) + \delta(I) \subseteq I$, and put $IA = \bigoplus_n It^n$. Clearly $R(IA) \subseteq IA$ and $(IA)t \subseteq IA$. Iterating (1) gives $t^n r \in \sum_{m \leq n} R t^m$, so $(it^n)r \in \sum_m It^m$ for $i \in I$ and $r \in R$. Finally $t(it^n) = \sigma(i)t^{n+1} + \delta(i)t^n \in IA$. Since R and t generate A , IA is an ideal of A . It is graded by (3), and $IA \cap R = I$. Hence $I \in \{0, R\}$ when A is gr-simple. If δ is inner, then $\delta = \delta_u$ with $u \in R_\gamma$ by Lemma 2.3(2), and Lemma 2.4 gives a nonzero proper graded ideal. $\square$

For $\sigma = \text{id}$ the converse holds in characteristic zero [1, Theorem 1.10(2)]. For $\sigma \neq \text{id}$ it fails.

Proposition 3.2. *In Example 2.5, $R = \mathbb{Q}[x]$ is (σ, δ) -gr-simple and δ is outer, but A is not gr-simple.*

Proof. The nonzero proper graded ideals of $\mathbb{Q}[x]$ are the ideals (x^m) with $m \geq 1$, and none is δ -stable since $\delta(x^m) = (2^m - 1)x^{m-1}$. As $R_{-1} = 0$, Lemma 2.3(2) shows that the only inner σ -derivation in this setting is 0, while $\delta(x) = 1$; so δ is outer.

Let $w = 1 + xt \in A_0$. From $tx = 2xt + 1$ we get

$$wx = x + x(2xt + 1) = 2xw, \quad tw = t + (2xt + 1)t = 2wt.$$

Since 2 is invertible, $wA = Aw$, and Aw is a nonzero graded ideal. The assignment $x \mapsto X$, $t \mapsto -X^{-1}$ kills $tx - 2xt - 1$, so it defines a homomorphism $A \rightarrow \mathbb{Q}[X^{\pm 1}]$. It kills w and is the identity on $\mathbb{Q}$, so $Aw \neq A$. $\square$

Remark 3.3. The element w comes from a localization. On the gr-simple ring $R' = \mathbb{Q}[x^{\pm 1}]$ the maps σ and δ extend by the same formulas, and $\delta = \delta_u$ with $u = -x^{-1} \in R'_{-1}$, since both are σ -derivations and $\delta_u(x) = -1 + 2 = 1 = \delta(x)$. By Lemma 2.4, $s = t + x^{-1}$ generates a proper graded ideal of $R'[t; \sigma, \delta]$, and $w = xs$. Thus δ becomes inner after inverting x . The element w is normal of t -degree one (see [18] for such elements and [17] for localizations of the quantized Weyl algebra). In physics, $tx = 2xt + 1$ is the relation $aa^\dagger - qa^\dagger a = 1$ of the q -oscillator of Arik and Coon [3] with $q = 2$, $a = t$ and $a^\dagger = x$; see also [20]. On the module with basis $e_0, e_1, \dots$ given by $xe_n = e_{n+1}$ and $te_n = (2^n - 1)e_{n-1}$, the element w acts by $we_n = 2^n e_n$, that is, as the operator q^N , where $Ne_n = ne_n$.

The leading-coefficient argument of [1] fails because, for $f = a_n t^n + a_{n-1} t^{n-1} + \dots$,

$$tf - ft = (\sigma(a_n) - a_n)t^{n+1} + (\delta(a_n) + \sigma(a_{n-1}) - a_{n-1})t^n + \dots$$

The commutator with t no longer lowers the t -degree, and ideals of leading coefficients need not be δ -stable. Here the leading coefficients of the elements of Aw of t -degree one, together with 0, form the ideal (x) : if $a_1 t + a_0 \in Aw$, applying the homomorphism above gives $a_1 = xa_0$, and conversely $a_0 w = a_0 + xa_0 t$. The ideal (x) is not δ -stable.

3.2. A characterization over gr-simple coefficient rings. From now on σ is an automorphism. By Lemma 2.3(3), $\sigma(R_\tau) = R_{\sigma(\tau)}$ for every τ . The proof of Theorem B rests on ideals of leading coefficients; for $\sigma = \text{id}$ and the trivial grading, compare [30, Lemma 4.13]. For a graded ideal J of A and $n \geq 0$ let

$$L_n(J) = \left\{ a \in R : at^n + \sum_{i < n} a_i t^i \in J \text{ for some } a_i \in R \right\}.$$

Lemma 3.4. $L_n(J)$ is a graded ideal of R .

Proof. $L_n(J)$ is closed under addition, since J is. Let $a \in L_n(J)$, witnessed by $f = at^n + \sum_{i < n} a_i t^i \in J$, and let $r \in R$. Then $rf \in J$ has t^n -coefficient ra , so $ra \in L_n(J)$. By (1), $t^i \sigma^{-n}(r) \in \sigma^{i-n}(r)t^i + \sum_{m < i} Rt^m$ for every $i \geq 0$. Hence $f\sigma^{-n}(r) \in J$ has t -degree at most n and t^n -coefficient ar , so $ar \in L_n(J)$.

For gradedness, write $a = \sum_\tau a_\tau$. By (3), the component of f of degree $\tau\gamma^n$ lies in J , has t -degree at most n , and has t^n -coefficient a_τ . So $a_\tau \in L_n(J)$ for every τ . $\square$

Lemma 3.5. For $n \geq 1$ and $r \in R$,

$$t^n r = \sigma^n(r)t^n + D_n(r)t^{n-1} + \sum_{m < n-1} Rt^m, \quad D_n = \sum_{j=0}^{n-1} \sigma^j \delta \sigma^{n-1-j}.$$

Proof. For $n = 1$ this is (1). Assume the formula for n and multiply it on the left by t . By (1),

$$t^{n+1}r = \sigma^{n+1}(r)t^{n+1} + (\delta\sigma^n(r) + \sigma D_n(r))t^n + \sum_{m < n} Rt^m,$$

since $t(D_n(r)t^{n-1}) = \sigma D_n(r)t^n + \delta D_n(r)t^{n-1}$ and the remaining terms have t -degree at most $n-1$. Finally $\delta\sigma^n + \sigma D_n = \sum_{j=0}^n \sigma^j \delta \sigma^{n-j} = D_{n+1}$. $\square$

The next lemma records basic properties of monic normal elements.

Lemma 3.6. Let $n \geq 1$ and let $f = t^n + at^{n-1} + \sum_{i < n-1} a_i t^i$ be homogeneous. Then $f \in A_{\gamma^n}$ and $a \in R_\gamma$. Moreover:

- (1) f is normal if and only if $fr = \sigma^n(r)f$ for all $r \in R$ and $tf - ft \in Rf$. In that case $tf = ft + (\sigma(a) - a)f$.
- (2) If f is normal, then $\Delta_n = \delta_{-a}$. In particular Δ_n is inner.
- (3) If f is normal, then Af is a nonzero proper graded ideal of A .

Proof. The t^n -coefficient 1 lies in R_e , so $f \in A_{\gamma^n}$ by (3), and then $a \in R_{\gamma^n \gamma^{1-n}} = R_\gamma$. Since f is monic, $\text{lc}(gf) = \text{lc}(g)$ for every nonzero $g \in A$, hence

$$\deg_t(gf) = \deg_t(g) + n \quad (g \neq 0). \quad (4)$$

(1) Suppose f is normal and let $r \in R$. Write $fr = gf$ with $g \in A$. By (4), $\deg_t g \leq 0$, so $g \in R$, and comparing t^n -coefficients via Lemma 3.5 gives $g = \sigma^n(r)$. Similarly $ft = hf$ with $\deg_t h = 1$; comparing t^{n+1} -coefficients gives $h = t + h_0$ with $h_0 \in R$, so $tf - ft = -h_0 f \in Rf$. Conversely, assume both conditions

and write $tf - ft = cf$ with $c \in R$. Then $fr = \sigma^n(r)f$ and $ft = (t-c)f$, so $fA \subseteq Af$ because R and t generate A . Since σ^n is bijective, $rf = f\sigma^{-n}(r)$ and $tf = ft + f\sigma^{-n}(c)$, so $Af \subseteq fA$. Finally $tf = t^{n+1} + \sigma(a)t^n + \dots$ and $ft = t^{n+1} + at^n + \dots$, and comparing t^n -coefficients in $tf - ft = cf$ gives $c = \sigma(a) - a$.

(2) By (1), $fr = \sigma^n(r)f$. Compare t^{n-1} -coefficients. By Lemma 3.5, t^nr contributes $D_n(r)$ and $at^{n-1}r$ contributes $a\sigma^{n-1}(r)$; the terms a_it^ir with $i < n-1$ do not contribute. On the right, $\sigma^n(r)f$ contributes $\sigma^n(r)a$. Hence

$$D_n(r) + a\sigma^{n-1}(r) = \sigma^n(r)a.$$

Substitute $r = \sigma^{1-n}(y)$. Since $D_n\sigma^{1-n} = \sum_{j=0}^{n-1} \sigma^j \delta \sigma^{-j} = \Delta_n$, this becomes

$$\Delta_n(y) = \sigma(y)a - ay = (-a)y - \sigma(y)(-a) = \delta_{-a}(y).$$

(3) $Af = fA$ is an ideal, graded because f is homogeneous, and nonzero. By (4) its nonzero elements have t -degree at least $n \geq 1$, so $1 \notin Af$. $\square$

Theorem 3.7. *Assume that R is gr-simple and σ is an automorphism. Then A is gr-simple if and only if A contains no monic normal element.*

Proof. If A contains a monic normal element, A is not gr-simple by Lemma 3.6(3). Conversely, let $J \neq 0$ be a graded ideal of A , and let n be the least t -degree of a nonzero element of J . By Lemma 3.4, $L_n(J)$ is a nonzero graded ideal of R , so $L_n(J) = R$. Choose $g \in J$ of t -degree n with t^n -coefficient 1. If $n = 0$, then $g = 1 \in J$ and $J = A$. Let $n \geq 1$. Since $1 \in R_e$, the component of g of degree γ^n lies in J and has the form

$$f = t^n + at^{n-1} + \sum_{i < n-1} a_i t^i.$$

For $r \in R$, both fr and $\sigma^n(r)f$ have t^n -coefficient $\sigma^n(r)$. So $fr - \sigma^n(r)f \in J$ has t -degree less than n , and it vanishes by minimality of n . Likewise, in $tf - ft - (\sigma(a) - a)f \in J$ the t^{n+1} -terms cancel and the t^n -coefficients agree, so this element vanishes. By Lemma 3.6(1), f is a monic normal element. Hence, if A contains no monic normal element, then $J = A$. $\square$

The same argument also determines the generators of graded ideals.

Corollary 3.8. *Assume that R is gr-simple and σ is an automorphism. Every nonzero graded ideal J of A has the form $J = Af = fA$ for a unique homogeneous normal polynomial f that is monic. For $J = A$ this polynomial is 1; for a proper ideal it has positive t -degree.*

Proof. Let n be the least t -degree of a nonzero element of J . If $n = 0$, the proof of Theorem 3.7 gives $J = A$, and we take $f = 1$. If $n \geq 1$, that proof constructs a homogeneous monic normal polynomial $f \in J$ of t -degree n .

Let $0 \neq g \in J$, with $\deg_t g = m \geq n$ and $\text{lc}(g) = a$. Since f is monic, $g - at^{m-n}f$ belongs to J and has t -degree less than m , unless it is zero. Repeating this step gives $g = qf + r$, where $r = 0$ or $\deg_t r < n$. As $r \in J$, minimality of n gives $r = 0$. Hence $J = Af = fA$.

If f' is another monic generator, write $f' = af$ and $f = bf'$ with $a, b \in A$. Multiplication on the right by a monic polynomial adds its t -degree, so these two equalities imply $\deg_t f' = \deg_t f$ and $a \in R$. Comparing leading coefficients gives $a = 1$, and thus $f' = f$. $\square$

For the trivial grading, Lam and Leroy give necessary and sufficient conditions for an Ore extension to be simple [21, Theorems 4.5 and 5.8]; see also [30]. The corollaries below give conditions on δ that exclude monic normal elements; for the trivial grading they are simplicity criteria over simple coefficient rings.

Corollary 3.9. *Assume that R is gr-simple and σ is an automorphism. If Δ_n is outer for every $n \geq 1$, then A is gr-simple.*

Proof. Combine Theorem 3.7 and Lemma 3.6(2). $\square$

The argument of Lemma 3.6(2) is the one in [10, proof of Lemma 4.8]. There the coefficient ring is a locally finite connected graded algebra generated in degree one, without zero divisors in degree one, whose only normal elements are the scalars, σ is a graded automorphism and δ is homogeneous of degree one; if no Δ_n is inner, then the only normal elements of the Ore extension are the scalars.

Corollary 3.9 never applies to $\sigma = \text{id}$ if $p1_R = 0$ for a prime p , since then $\Delta_p = p\delta = 0$. Theorem 3.7 has no such restriction, and for $\sigma = \text{id}$ its argument works under a weaker hypothesis on R .

Corollary 3.10. *Let $\sigma = \text{id}$, let (id, δ) be γ -compatible, and assume that R is (id, δ) -gr-simple. Then $R[t; \delta]$ is gr-simple if and only if it contains no homogeneous central element that is monic of positive t -degree.*

Proof. A central element is normal, so one direction is Lemma 3.6(3). Conversely, let $J \neq 0$ be a graded ideal and let n be the least t -degree of a nonzero element of J . By Lemma 3.4, $L_n(J)$ is a nonzero graded ideal of R . If $f = \sum_i a_i t^i \in J$ with $\deg_t f \leq n$, then $tf - ft = \sum_i \delta(a_i) t^i \in J$, so $\delta(L_n(J)) \subseteq L_n(J)$. Hence $L_n(J) = R$. If $n = 0$, then $1 \in J$. Let $n \geq 1$. As in the proof of Theorem 3.7, J contains a homogeneous element $f = t^n + \sum_{i < n} a_i t^i$. For $r \in R$, the element $fr - rf \in J$ has t -degree less than n , and so does $tf - ft = \sum_{i < n} \delta(a_i) t^i$, because $\delta(1) = 0$. Both vanish by minimality of n , so f is central. $\square$

Corollary 3.10 is the graded form of [30, Lemma 4.13 and Proposition 4.14]. It needs no hypothesis on the centre of $R[t; \delta]$, in contrast with the centre criterion [1, Theorem 1.10(3)], which assumes that this centre is a graded subring (this is automatic when Γ is abelian). Compare also [30, Theorem 4.15]: $R[t; \delta]$ is simple if and only if R is δ -simple and the centre of $R[t; \delta]$ is a field.

Corollary 3.11. *Assume that R is gr-simple with $\text{char } R = 0$, that σ is an automorphism, and that $\sigma\delta\sigma^{-1} - \delta$ is inner. Then A is gr-simple if and only if δ is outer. This applies in particular when $\sigma\delta = \delta\sigma$.*

Proof. Necessity is Proposition 3.1. For sufficiency, write $\sigma\delta\sigma^{-1} - \delta = \delta_v$. The pair $(\sigma, \sigma\delta\sigma^{-1} - \delta)$ is γ -compatible by Lemma 2.3(3), so we may take $v \in R_\gamma$ by Lemma 2.3(2).

We show by induction that $\sigma^j \delta \sigma^{-j} - \delta = \delta_{v_j}$ with $v_j \in R_\gamma$ for all $j \geq 0$. For $j = 0$ take $v_0 = 0$. If it holds for j , then, since $u \mapsto \delta_u$ is additive and $\sigma\delta_{v_j}\sigma^{-1} = \delta_{\sigma(v_j)}$ with $\sigma(v_j) \in R_\gamma$ by Lemma 2.3(3),

$$\sigma^{j+1} \delta \sigma^{-j-1} - \delta = \sigma(\sigma^j \delta \sigma^{-j} - \delta) \sigma^{-1} + (\sigma\delta\sigma^{-1} - \delta) = \delta_{\sigma(v_j) + v}.$$

Summing over $j = 0, \dots, n-1$ gives $\Delta_n = n\delta + \delta_{w_n}$ with $w_n = \sum_{j < n} v_j \in R_\gamma$.

Let $n \geq 1$. The element $n1_R$ is central, homogeneous of degree e , and nonzero since $\text{char } R = 0$. So $(n1_R)R$ is a nonzero graded ideal, hence equal to R , and $n1_R$ is invertible with central inverse n^{-1} . If $\Delta_n = \delta_y$ for some $y \in R$, then $n\delta = \delta_{y-w_n}$, and multiplying by n^{-1} gives $\delta = \delta_{n^{-1}(y-w_n)}$. This contradicts the outerness of δ . Hence every Δ_n is outer, and Corollary 3.9 applies. $\square$

Example 3.12 (Characteristic p). Let $\text{char } k = p > 0$ and let $R = k[x^{\pm 1}]$ be $\mathbb{Z}$ -graded with $\deg x = 1$. Every nonzero homogeneous element of R is a unit, so R is gr-simple. Let $\delta = \frac{d}{dx}$. Then $\delta(R_n) \subseteq R_{n-1}$, so $A = R[t; \delta]$ is graded with $\deg t = -1$ by Theorem 2.1. Since $\binom{p}{i} = 0$ in k for $0 < i < p$, we have

$$t^p r = \sum_{i=0}^p \binom{p}{i} \delta^i(r) t^{p-i} = rt^p + \delta^p(r) \quad (r \in R).$$

Moreover $\delta^p(x^n) = n(n-1) \cdots (n-p+1)x^{n-p} = 0$ for all $n \in \mathbb{Z}$, because the product of p consecutive integers is divisible by p . Hence t^p is a homogeneous central element that is monic of t -degree p , and A is not gr-simple by Lemma 3.6(3). In characteristic zero the same formulas give a gr-simple ring: R is commutative, so every inner derivation of R vanishes, $\delta \neq 0$ is outer, and Corollary 3.11 applies.

Remark 3.13. Let R be prime, σ an automorphism and δ an X -outer σ -derivation, that is, δ is not δ_b with b in the symmetric Martindale ring of quotients of R . Chuang and Tsai [8] study semi-invariant polynomials, with coefficients in the symmetric Martindale ring of quotients. They show that, in characteristic zero, the minimal positive degree of a monic semi-invariant polynomial, when one exists, is the least integer $\nu \geq 1$ for which

$$\sigma^\nu \delta \sigma^{-\nu} - \delta$$

is X -inner. The hypothesis of Corollary 3.11 corresponds to $\nu = 1$. There is no monic semi-invariant polynomial $t + b$ of degree one: if $(t + b)r = r'(t + b)$ for each r , then comparing t -coefficients gives $r' = \sigma(r)$, and comparing constant terms gives $\delta(r) = \sigma(r)b - br$, so δ would be δ_{-b} , hence X -inner. For the trivial grading and R simple, Corollary 3.11 is therefore close to their result. It does not require R prime: kC_2 is gr-simple but not prime (see Section 1).

Remark 3.14. We do not know whether the innerness hypothesis in Corollary 3.11 can be removed. The proof needs it for the following reason. Suppose that $\sigma^2 = \text{id}$, $\sigma\delta\sigma^{-1} = -\delta$, and $\delta^2(r) = rb - br$ for all $r \in R$, where $b \in R_{\gamma_2}$ satisfies $\sigma(b) = b$ and $\delta(b) = 0$. Then $D_2 = \delta\sigma + \sigma\delta = 0$ in Lemma 3.5, so $t^2r = rt^2 + \delta^2(r)$. The homogeneous element $f = t^2 + b$ satisfies $fr = rf$ and $tf - ft = (\sigma(b) - b)t + \delta(b) = 0$, so A is not

gr-simple by Lemma 3.6(3). On the other hand, if R is gr-simple of characteristic zero, then $2 \cdot 1_R$ is a central unit (see the proof of Corollary 3.11), so $\sigma\delta\sigma^{-1} - \delta = -2\delta$ is outer whenever δ is. We do not know of such data with R gr-simple of characteristic zero and δ outer. By Remark 2.2(3) and [1, Theorem 1.10(2)], σ would have to differ from conjugation by a unit of R_γ .

3.3. Group algebras. In this subsection we use the notation of Definition 1.2. Taking $\tau = \rho = e$ in the cocycle identity gives $c(e) = 0$, and then $c(\tau^{-1}) = -\chi(\tau)^{-1}c(\tau)$.

Proposition 3.15. *Let G be a group, $R = kG$ with $R_\tau = k\tau$, $\gamma \in G$, and $\chi: G \rightarrow k^\times$ a homomorphism. Put $\sigma(\tau) = \chi(\tau)\gamma\tau\gamma^{-1}$.*

- (1) σ extends to a k -algebra automorphism of kG with $\sigma(R_\tau) = R_{\gamma\tau\gamma^{-1}}$.
- (2) The k -linear σ -derivations δ for which (σ, δ) is γ -compatible are exactly the maps $\delta_c(\tau) = c(\tau)\gamma\tau$ with $c \in Z^1(G, k_\chi)$.
- (3) δ_c is inner if and only if $c \in B^1(G, k_\chi)$.
- (4) $\sigma\delta_c\sigma^{-1} - \delta_c = -\delta_u$ with $u = c(\gamma)\gamma \in R_\gamma$. In particular it is inner.

Proof. (1) For $\tau, \rho \in G$,

$$\sigma(\tau)\sigma(\rho) = \chi(\tau)\chi(\rho)\gamma\tau\gamma^{-1}\gamma\rho\gamma^{-1} = \sigma(\tau\rho).$$

So $\tau \mapsto \sigma(\tau)$ is a group homomorphism into the units of kG , and it extends k -linearly to an algebra endomorphism of kG . Since $k^\times$ is abelian, $\chi(\gamma^{-1}\tau\gamma) = \chi(\tau)$, and $\psi(\tau) = \chi(\tau)^{-1}\gamma^{-1}\tau\gamma$ satisfies $\sigma\psi(\tau) = \tau = \psi\sigma(\tau)$. Hence $\psi = \sigma^{-1}$. Finally, $\sigma(k\tau) = k\gamma\tau\gamma^{-1}$ because $\chi(\tau) \neq 0$.

- (2) Compatibility forces $\delta(\tau) \in R_{\gamma\tau} = k\gamma\tau$, say $\delta(\tau) = c(\tau)\gamma\tau$. Then

$$\sigma(\tau)\delta(\rho) + \delta(\tau)\rho = \chi(\tau)\gamma\tau\gamma^{-1}c(\rho)\gamma\rho + c(\tau)\gamma\tau\rho = (c(\tau) + \chi(\tau)c(\rho))\gamma\tau\rho.$$

Since G is a k -basis of kG , the identity $\delta(\tau\rho) = \sigma(\tau)\delta(\rho) + \delta(\tau)\rho$ is exactly the cocycle identity. Conversely, for a cocycle c the k -linear map δ_c satisfies the Leibniz rule on the basis G , hence everywhere by bilinearity.

- (3) By Lemma 2.3(2), δ_c is inner if and only if $\delta_c = \delta_u$ for some $u \in R_\gamma$. Write $u = x\gamma$ with $x \in k$. Then

$$\delta_u(\tau) = x\gamma\tau - \chi(\tau)\gamma\tau\gamma^{-1}x\gamma = x(1 - \chi(\tau))\gamma\tau.$$

So δ_c is inner if and only if $c = -x(\chi - 1)$ for some $x \in k$, that is, $c \in B^1(G, k_\chi)$.

- (4) We have $\sigma^{-1}(\tau) = \chi(\tau)^{-1}\gamma^{-1}\tau\gamma$ and $\sigma(\tau\gamma) = \chi(\tau)\chi(\gamma)\gamma\tau$. Hence

$$\sigma\delta_c\sigma^{-1}(\tau) = \chi(\tau)^{-1}c(\gamma^{-1}\tau\gamma)\sigma(\tau\gamma) = \chi(\gamma)c(\gamma^{-1}\tau\gamma)\gamma\tau.$$

The cocycle identity, applied to $\gamma^{-1} \cdot (\tau\gamma)$ and to $\tau \cdot \gamma$, together with $c(\gamma^{-1}) = -\chi(\gamma)^{-1}c(\gamma)$, gives

$$\chi(\gamma)c(\gamma^{-1}\tau\gamma) = -c(\gamma) + c(\tau) + \chi(\tau)c(\gamma) = c(\tau) - (1 - \chi(\tau))c(\gamma).$$

By the formula in (3) with $x = c(\gamma)$, this reads $\sigma\delta_c\sigma^{-1}(\tau) = \delta_c(\tau) - \delta_u(\tau)$ with $u = c(\gamma)\gamma$. $\square$

Every ι_γ -graded k -algebra endomorphism σ of kG has the form $\tau \mapsto \chi(\tau)\gamma\tau\gamma^{-1}$ for a character χ : the element $\sigma(\tau) \in k\gamma\tau\gamma^{-1}$ is nonzero because $\sigma(\tau)\sigma(\tau^{-1}) = 1$, and multiplicativity makes χ a homomorphism. Hence Proposition 3.15 describes all γ -compatible pairs of k -linear maps on kG . If $\chi = 1$, then σ is conjugation by the homogeneous unit γ , and Remark 2.2(3) identifies $kG[t; \sigma, \delta_c]$ with the differential polynomial ring $kG[t'; \delta']$, where $\deg t' = e$, $\delta'(\tau) = c(\tau)\tau$ and $c \in \text{Hom}(G, k)$. If $\chi \neq 1$, then σ is not conjugation by a homogeneous unit: the homogeneous units are the elements $\lambda\rho$ with $\lambda \in k^\times$, $\rho \in G$, and conjugation by $\lambda\rho$ maps τ to $\rho\tau\rho^{-1}$, whereas $\sigma(\tau) = \chi(\tau)\gamma\tau\gamma^{-1}$.

The ring kG is gr-simple, because every nonzero homogeneous element is a unit. By Proposition 3.15(4), Corollary 3.11 therefore applies to the family of Proposition 3.15.

Corollary 3.16. *Let $\text{char } k = 0$, and let γ, χ, σ and $c \in Z^1(G, k_\chi)$ be as in Proposition 3.15. Then $kG[t; \sigma, \delta_c]$, graded by $\deg t = \gamma$, is gr-simple if and only if $[c] \neq 0$ in $H^1(G, k_\chi)$.*

Proof. Combine Corollary 3.11 with Proposition 3.15(3),(4). $\square$

The cohomological criterion can be tested on generators.

Corollary 3.17. *In the setting of Corollary 3.16, suppose that G is generated by $g_1, \dots, g_m$. If $\chi = 1$, then $kG[t; \sigma, \delta_c]$ is gr-simple if and only if $c(g_i) \neq 0$ for some i . If $\chi \neq 1$, it is gr-simple if and only if*

$$c(g_i)(\chi(g_j) - 1) - c(g_j)(\chi(g_i) - 1) \neq 0 \quad \text{for some } i, j.$$

Proof. By the cocycle identity and $c(g^{-1}) = -\chi(g)^{-1}c(g)$, two cocycles that agree on the generators agree on G . So $[c] = 0$ if and only if $c(g_i) = x(\chi(g_i) - 1)$ for all i and some $x \in k$. If $\chi = 1$, this means $c(g_i) = 0$ for all i . If $\chi \neq 1$, some $\chi(g_i) - 1$ is nonzero, and the vector $(c(g_i))_i$ is a multiple of $(\chi(g_i) - 1)_i$ if and only if all the displayed 2×2 minors vanish. Now apply Corollary 3.16. $\square$

For a presentation $G = \langle g_1, \dots, g_m \mid w_1, \dots, w_r \rangle$, the possible values $(c(g_i))_i$ are the solutions of a linear system. Let F be the free group on $g_1, \dots, g_m$, let $\pi: F \rightarrow G$ be the projection and $\tilde{\chi} = \chi \circ \pi$. Every vector $(u_i) \in k^m$ is $(\tilde{c}(g_i))_i$ for a unique cocycle $\tilde{c} \in Z^1(F, k_{\tilde{\chi}})$. This cocycle factors through G if and only if $\tilde{c}(w_j) = 0$ for every relator. Indeed, $\tilde{\chi}(w_j) = 1$, so $\tilde{c}(gw_jg^{-1}) = \tilde{\chi}(g)\tilde{c}(w_j)$ for every $g \in F$; hence $\tilde{c}$ vanishes on the normal closure N of the relators, and $\tilde{c}(gn) = \tilde{c}(g)$ for $n \in N$. Corollary 3.17 then tests gr-simplicity on the solution space.

Remark 3.18. If $\chi(z) \neq 1$ for some z in the centre of G , then $H^1(G, k_\chi) = 0$. Indeed, applying the cocycle identity to $z\tau = \tau z$ gives $c(z) + \chi(z)c(\tau) = c(\tau) + \chi(\tau)c(z)$, that is, $c(\tau) = x(\chi(\tau) - 1)$ with $x = c(z)(\chi(z) - 1)^{-1}$. So every cocycle is a coboundary. In particular, if G is abelian and $\chi \neq 1$, then δ_c is inner and no ring in the family is gr-simple (Proposition 3.1). Hence, if $\chi \neq 1$ and $kG[t; \sigma, \delta_c]$ is gr-simple, then G is nonabelian and χ is trivial on the centre of G .

Example 3.19. Let $G = F(a, b)$ be the free group on a, b , let $k = \mathbb{Q}$, fix $q \in \mathbb{Q}^\times$ with $q \neq 1$, and take $\gamma = a$, $\chi(a) = q$, $\chi(b) = 1$. A cocycle on a free group is determined by arbitrary values on the free generators; let c be the cocycle with $c(a) = 0$ and $c(b) = 1$. From $tr = \sigma(r)t + \delta_c(r)$ we get

$$ta = \chi(a)aaa^{-1}t + c(a)aa = qat, \quad tb = \chi(b)aba^{-1}t + c(b)ab = aba^{-1}t + ab.$$

So the ring is generated by $\mathbb{Q}F(a, b)$ and t subject to these relations, with $\deg t = a$. Every coboundary $x(\chi - 1)$ vanishes at b , so $[c] \neq 0$, and the ring is gr-simple by Corollary 3.16. More precisely, cocycles correspond to pairs $(c(a), c(b)) \in \mathbb{Q}^2$ and coboundaries to the pairs $(x(q - 1), 0)$; hence $H^1(F(a, b), \mathbb{Q}_\chi) \cong \mathbb{Q}$ via $[c] \mapsto c(b)$, and the ring attached to c is gr-simple exactly when $c(b) \neq 0$. Since $c(\gamma) = c(a) = 0$, Proposition 3.15(4) gives $\sigma\delta_c = \delta_c\sigma$, the commuting case of Corollary 3.11. Here $\sigma(R_b) = R_{aba^{-1}} \neq R_b$, and since a does not commute with b , no differential polynomial ring over $\mathbb{Q}F(a, b)$ admits a compatible grading with $\deg t = a$ (Remark 2.2(2)). The degree can nevertheless be moved to e by Remark 2.2(3) with $g = a$.

3.4. Group algebras as skew group rings. For group algebras every $R_\gamma = k\gamma$ contains the unit γ . So, by Remark 2.2(3), the degree of the Ore variable can be moved to e , and the extension becomes a skew group ring over a polynomial ring. In particular, the noncentral degree of t in Example 3.19 comes from the choice of the variable t .

Keep the notation of Proposition 3.15 and put $A = kG[t; \sigma, \delta_c]$. For $\tau \in G$ let

$$\varphi_\tau(X) = \chi(\tau)X + c(\tau) \in k[X].$$

Since $\varphi_\tau(\varphi_\rho(X)) = \chi(\tau\rho)X + c(\tau) + \chi(\tau)c(\rho)$, the cocycle identity says exactly that $\varphi_{\tau\rho} = \varphi_\tau \circ \varphi_\rho$. Thus $\Phi: \tau \mapsto \varphi_\tau$ is a homomorphism from G to $\text{Aff}(1, k)$, and every homomorphism $G \rightarrow \text{Aff}(1, k)$ arises in this way from a unique pair (χ, c) with $c \in Z^1(G, k_\chi)$. Moreover, $c = x(\chi - 1)$ if and only if $\varphi_\tau(-x) = -x$ for all τ ; hence $[c] = 0$ if and only if $\Phi(G)$ has a fixed point in k . Let G act on $k[X]$ by the k -algebra automorphisms $\theta_\tau(f) = f \circ \varphi_\tau^{-1}$, and let $k[X] \rtimes_\theta G = \bigoplus_\tau k[X]u_\tau$ be the corresponding skew group ring, so that $u_\tau f = \theta_\tau(f)u_\tau$ and $\deg u_\tau = \tau$.

Proposition 3.20. Put $s = \gamma^{-1}t \in A_e$. Then $s\tau = \tau\varphi_\tau(s)$ for $\tau \in G$, $A_e = k[s]$ is a polynomial ring in s , and $A_\tau = \tau k[s]$ for every $\tau \in G$. The map $f(X)u_\tau \mapsto f(s)\tau$ is an isomorphism of G -graded k -algebras $k[X] \rtimes_\theta G \rightarrow A$.

Proof. Since $\chi(\tau)$ and $c(\tau)$ are central scalars,

$$s\tau = \gamma^{-1}(\chi(\tau)\gamma\tau\gamma^{-1}t + c(\tau)\gamma\tau) = \tau(\chi(\tau)s + c(\tau)) = \tau\varphi_\tau(s).$$

By (1), $t\gamma^{-n} \in \chi(\gamma)^{-n}\gamma^{-n}t + R$, so by induction $s^n = \lambda_n\gamma^{-n}t^n + (\text{lower terms})$ with $\lambda_n \in k^\times$. Since $A_e = \bigoplus_n k\gamma^{-n}t^n$ by (3) and $s^n \in A_e$, the powers s^n form a k -basis of A_e . Since τ is a unit in A_τ , $\tau A_e \subseteq A_\tau$ and $\tau^{-1}A_\tau \subseteq A_e$, so $A_\tau = \tau k[s]$. From $\tau^{-1}s\tau = \varphi_\tau(s)$ we get $\tau f(s)\tau^{-1} = f(\varphi_\tau^{-1}(s)) = \theta_\tau(f)(s)$. Hence

$$f(s)\tau \cdot g(s)\rho = f(s)\theta_\tau(g)(s)\tau\rho,$$

which is the image of $(fu_\tau)(gu_\rho) = f\theta_\tau(g)u_{\tau\rho}$. So the map is multiplicative, and it maps $k[X]u_\tau$ bijectively onto $k[s]\tau = \tau k[s] = A_\tau$. $\square$

Skew group rings of polynomial rings also appear in the study of iterated Hopf–Ore extensions of group algebras [14].

Theorem 3.21. *Let $A = kG[t; \sigma, \delta_c]$ and $\Phi = (\chi, c)$ be as above, and let $\bar{k}$ be an algebraic closure of k .*

- (1) *The graded ideals of A are the ideals $Af(s)$, where $f = 0$ or $f \in k[X]$ is monic with $f \circ \varphi_\tau \in k^\times f$ for all $\tau \in G$. Distinct f give distinct ideals.*
- (2) *A is gr-simple if and only if $\Phi(G)$ has no finite orbit in $\bar{k}$, if and only if $[c]$ restricts to a nonzero class in $H^1(H, k_\chi)$ for every subgroup H of finite index in G .*
- (3) *If $\text{char } k = 0$, then A is gr-simple if and only if $[c] \neq 0$ in $H^1(G, k_\chi)$.*
- (4) *A is simple if and only if it is gr-simple and Φ is injective. In particular, if A is simple, then G is metabelian.*

Proof. (1) Let J be a graded ideal and $I = J \cap A_e$, an ideal of $A_e = k[s]$. For $\tau \in G$, $\tau I \subseteq J_\tau$ and $\tau^{-1}J_\tau \subseteq I$, so $J = \bigoplus_\tau \tau I$; moreover $\tau I \tau^{-1} \subseteq J \cap A_e = I$. Conversely, if I is an ideal of $k[s]$ with $\tau I \tau^{-1} \subseteq I$ for all τ , then $\bigoplus_\tau \tau I$ is an ideal of A , because $\rho g(s)\tau I = \rho\tau(\tau^{-1}g(s)\tau)I \subseteq \rho\tau I$ and $\tau I \rho g(s) = \tau\rho(\rho^{-1}I\rho)g(s) \subseteq \tau\rho I$. Thus $J \mapsto J \cap A_e$ is a bijection onto the ideals I of $k[s]$ with $\tau I \tau^{-1} \subseteq I$ for all τ . Write $I = f(s)k[s]$ with $f = 0$ or f monic. As $\tau f(s)\tau^{-1} = f(\varphi_\tau^{-1}(s))$ and $f \circ \varphi_\tau^{-1}$ has the same X -degree as f , the condition reads $f \circ \varphi_\tau^{-1} \in k^\times f$ for all τ , equivalently $f \circ \varphi_\tau \in k^\times f$ for all τ . Finally $\bigoplus_\tau \tau I = Af(s)$.

(2) By (1), A is gr-simple if and only if every monic f with $f \circ \varphi_\tau \in k^\times f$ for all τ equals 1. Let f be such a polynomial of positive degree, and let $Z \subseteq \bar{k}$ be its set of roots. If $x \in Z$, then $f \circ \varphi_\tau$ vanishes at $\varphi_\tau^{-1}(x)$, hence so does f . Thus Z is a finite nonempty $\Phi(G)$ -stable set, and its orbits are finite. Conversely, let $O \subseteq \bar{k}$ be a finite orbit, and let M be the finite set of minimal polynomials over k of the elements of O . For $x \in O$ and $\tau \in G$, $m_x \circ \varphi_\tau$ is irreducible of degree $\deg m_x$, because φ_τ is an affine bijection defined over k , and it vanishes at $\varphi_\tau^{-1}(x) \in O$; hence $m_x \circ \varphi_\tau \in k^\times m_{\varphi_\tau^{-1}(x)}$. The map $m_x \mapsto m_{\varphi_\tau^{-1}(x)}$ is well defined on M and surjective, since φ_τ^{-1} permutes O ; so it is a bijection of M . Hence $f = \prod_{m \in M} m$ is monic of positive degree with $f \circ \varphi_\tau \in k^\times f$ for all τ .

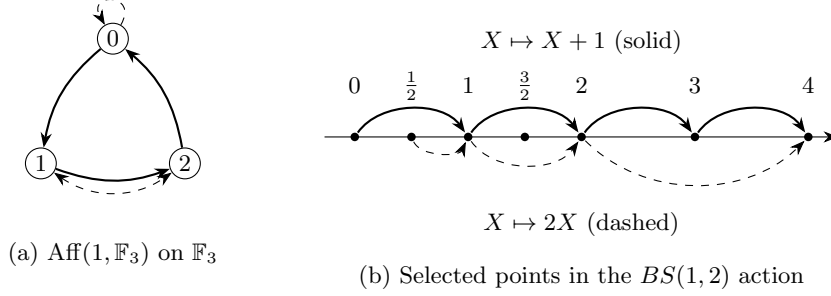
For the second equivalence, a finite orbit exists if and only if some subgroup H of finite index fixes a point $x \in \bar{k}$: take for H the stabilizer of a point of the orbit; conversely the orbit of x has at most $[G : H]$ elements. If H fixes x , then $c(h) = x(1 - \chi(h))$ for $h \in H$. If $\chi|_H \neq 1$, choosing $h_0 \in H$ with $\chi(h_0) \neq 1$ gives $x = c(h_0)/(1 - \chi(h_0)) \in k$, so $c|_H = (-x)(\chi - 1)$; if $\chi|_H = 1$, then $c|_H = 0$. In both cases the restriction of $[c]$ to H is zero. Conversely, if $c|_H = y(\chi - 1)$ with $y \in k$, then H fixes $-y$.

(3) Let $O \subseteq \bar{k}$ be a finite orbit, and put $b = |O|^{-1} \sum_{x \in O} x$, which is defined because $\text{char } k = 0$. Each φ_τ is affine and permutes O , so $\varphi_\tau(b) = b$. The argument of (2) with $H = G$ gives $[c] = 0$. Conversely, if $[c] = 0$, then $\Phi(G)$ fixes a point of k . Now apply (2).

(4) By Proposition 3.20, A is the skew group ring $k[X] \rtimes_\theta G$ over the commutative ring $k[X]$. By [28, Theorem 6.13], A is simple if and only if $k[X]$ is G -simple (that is, 0 and $k[X]$ are its only G -stable ideals) and $k[X]$ is maximal commutative in A . By the proof of (1), $k[X]$ is G -simple if and only if A is gr-simple. An element $\sum_\tau f_\tau u_\tau$ commutes with every $g \in k[X]$ if and only if $f_\tau(\theta_\tau(g) - g) = 0$ for all τ and g . As $k[X]$ is a domain, $k[X]$ is maximal commutative if and only if $\theta_\tau \neq \text{id}$ for $\tau \neq e$, that is, if and only if Φ is injective. Then $G \cong \Phi(G) \subseteq \text{Aff}(1, k) = k \rtimes k^\times$, which is metabelian. $\square$

Part (3) recovers Corollary 3.16 without using Corollary 3.11. The cohomological criterion fails in positive characteristic.

The following two affine actions illustrate the orbit criterion in Theorem 3.21. In panel (a), corresponding to Example 3.22, solid arrows represent $X \mapsto X + 1$ and dashed arrows represent $X \mapsto -X$. No point is fixed by the whole group, but $\mathbb{F}_3$ is a finite orbit, and $X^3 - X$ gives the graded ideal $A(s^3 - s)$. In panel (b), corresponding to Example 3.25(2), solid arrows represent $X \mapsto X + 1$ and dashed arrows represent $X \mapsto 2X$; only selected points and images are shown. Every orbit in $\bar{\mathbb{Q}}$ contains $x + \mathbb{Z}$ for any point x in that orbit, and is therefore infinite. Since G is defined as a subgroup of $\text{Aff}(1, \mathbb{Q})$, the action is faithful and the extension is simple.



Example 3.22. Let $k = \mathbb{F}_3$, $G = \text{Aff}(1, \mathbb{F}_3) \cong S_3$, and let Φ be the inclusion; thus $\chi(\varphi) = a$ and $c(\varphi) = b$ for $\varphi(X) = aX + b$. No point of $\mathbb{F}_3$ is fixed by G , so $[c] \neq 0$ in $H^1(G, k_\chi)$. However, $f = X^3 - X$ satisfies $f(aX + b) = a^3X^3 + b^3 - aX - b = af(X)$ for $a, b \in \mathbb{F}_3$. By Theorem 3.21(1), $A(s^3 - s)$ is a nonzero proper graded ideal of A for every choice of $\gamma \in G$. So Corollary 3.16 fails in characteristic 3.

More generally, the cohomological criterion fails for every finite group with $H^1(G, k_\chi) \neq 0$.

Corollary 3.23. *If G is finite, then $kG[t; \sigma, \delta_c]$ is not gr-simple, for every γ, χ and c . For example, let $\text{char } k = p > 0$, $G = C_p = \langle g \rangle$, $\chi = 1$, $\gamma \in G$ arbitrary and $c(g^i) = i \cdot 1_k$. Then $[c] \neq 0$ in $H^1(G, k)$, but $A(s^p - s)$, with $s = \gamma^{-1}t$ as in Proposition 3.20, is a nonzero proper graded ideal of A .*

Proof. Every $\Phi(G)$ -orbit in $\bar{k}$ has at most $|G|$ elements; apply Theorem 3.21(2). For $G = C_p$ and $\chi = 1$, the map c is a well-defined homomorphism because $p \cdot 1_k = 0$; moreover $B^1(G, k) = 0$ and $c \neq 0$, so $[c] \neq 0$. Since $c(\tau) \in \mathbb{F}_p$, the polynomial $f = X^p - X$ satisfies $f \circ \varphi_\tau = f$ for all τ , and Theorem 3.21(1) applies. $\square$

In characteristic zero this agrees with Corollary 3.16, since $H^1(G, k_\chi) = 0$ for finite G . In the example $\chi = 1$, so A is a differential polynomial ring by Remark 2.2(3). Thus the cohomological criterion already fails in the differential case.

Remark 3.24. Let $s = \gamma^{-1}t$, $\lambda \in k$, $G_\lambda = \{\tau \in G : \varphi_\tau(\lambda) = \lambda\}$ and $L_\lambda = A(s - \lambda)$. Then L_λ is a two-sided ideal if and only if λ is fixed by $\Phi(G)$. Indeed, by Remark 2.2(3) with $g = \gamma$, $A = kG[s; \sigma', \delta']$ with $\sigma'(\tau) = \chi(\tau)\tau$ and $\delta'(\tau) = c(\tau)\tau$, so

$$(s - \lambda)\tau = \sigma'(\tau)(s - \lambda) + (\varphi_\tau(\lambda) - \lambda)\tau \quad (\tau \in G).$$

Let D be the k -linear map with $D(\tau) = (\varphi_\tau(\lambda) - \lambda)\tau$. Since $s - \lambda$ is monic, $A = kG \oplus L_\lambda$. For $a = r + g(s - \lambda)$ with $r \in kG$, we have $L_\lambda a \subseteq L_\lambda$ if and only if $(s - \lambda)r \in L_\lambda$, that is, $D(r) \in kG \cap L_\lambda = 0$, that is, $r \in kG_\lambda$. So the idealizer $\{a \in A : L_\lambda a \subseteq L_\lambda\}$ equals $kG_\lambda \oplus L_\lambda$, and L_λ is two-sided if and only if $G_\lambda = G$.

If $\text{char } k = 0$ and A is not gr-simple, then $[c] = 0$ by Theorem 3.21(3), so $\Phi(G)$ fixes some $\lambda \in k$ and L_λ is a nonzero proper graded ideal. In Example 3.22 no such λ exists: every $\lambda \in \mathbb{F}_3$ has stabilizer $\{X \mapsto aX + \lambda(1 - a) : a = \pm 1\}$ of order 2, so no L_λ is two-sided, whereas $s^3 - s = \prod_{\lambda \in \mathbb{F}_3} (s - \lambda)$ generates a proper graded ideal. The obstruction is carried by the orbit $\mathbb{F}_3$ rather than by a point.

Example 3.25. (1) In Example 3.19, $\Phi(a)(X) = qX$ and $\Phi(b)(X) = X + 1$. The orbit of any $x \in \bar{k}$ contains $x + \mathbb{Z}$, so the ring is gr-simple, as already shown. Since $F(a, b)$ is not solvable and $\text{Aff}(1, \mathbb{Q})$ is metabelian, Φ is not injective, and the ring is not simple by Theorem 3.21(4).

(2) Let G be the subgroup of $\text{Aff}(1, \mathbb{Q})$ generated by $a(X) = 2X$ and $b(X) = X + 1$, let Φ be the inclusion, and let $\gamma = a$. Then $aba^{-1} = b^2$, and G is isomorphic to the Baumslag–Solitar group $BS(1, 2) = \langle a, b \mid aba^{-1} = b^2 \rangle$. Here $\chi(a) = 2$, $\chi(b) = 1$, $c(a) = 0$, $c(b) = 1$, and the orbit of every $x \in \mathbb{Q}$ contains $x + \mathbb{Z}$. Since $\sigma(b) = aba^{-1} = b^2$, the ring $\mathbb{Q}G[t; \sigma, \delta_c]$ is given by

$$ta = 2at, \quad tb = b^2t + ab, \quad \deg t = a,$$

and it is simple by Theorem 3.21(2),(4).

Example 3.26 (Translations). Let $\text{char } k = 0$, $G = \mathbb{Z}^2$ with basis g_1, g_2 , $\chi = 1$, and $\gamma \in G$. Then $\sigma = \text{id}$, since G is abelian, and $Z^1(G, k_1) = H^1(G, k) = \text{Hom}(G, k)$. For $c \in \text{Hom}(G, k)$ put $c_i = c(g_i)$. The map Φ sends τ to the translation $X \mapsto X + c(\tau)$. By Theorem 3.21(3), $kG[t; \delta_c]$ is gr-simple if and only if $c \neq 0$. By Theorem 3.21(4), it is simple if and only if c is injective, that is, if and only if c_1 and c_2 are linearly independent over $\mathbb{Q}$. For $k = \mathbb{R}$ and $(c_1, c_2) = (1, \sqrt{2})$ we obtain a simple ring. For $k = \mathbb{Q}$ and $(c_1, c_2) = (1, 2)$ the ring is gr-simple but not simple. For $G = \mathbb{Z}$ and $c \neq 0$ the ring is always simple.

4. POISSON POLYNOMIAL RINGS

A *Poisson algebra* is a commutative k -algebra P with a Lie bracket $\{-, -\}$ that is a derivation in each variable. It is Γ -graded if $P_\rho P_\tau \subseteq P_{\rho\tau}$ and $\{P_\rho, P_\tau\} \subseteq P_{\rho\tau}$ for all ρ, τ . A *Poisson ideal* is an ideal I with $\{P, I\} \subseteq I$, and P is *Poisson gr-simple* if 0 and P are its only graded Poisson ideals. An element $f \in P$ is *Poisson normal* if $\{P, f\} \subseteq fP$; then fP is a Poisson ideal.

A *Poisson derivation* of P is a derivation α with $\alpha(\{a, b\}) = \{\alpha(a), b\} + \{a, \alpha(b)\}$. Given α , a *Poisson α -derivation* is a derivation δ with

$$\delta(\{a, b\}) = \{\delta(a), b\} + \{a, \delta(b)\} + \alpha(a)\delta(b) - \delta(a)\alpha(b) \quad (a, b \in P).$$

For $u \in P$ put $\delta_u^p(a) = \{u, a\} - u\alpha(a)$. We say that δ is α -inner if $\delta = \delta_u^p$ for some $u \in P$. Finally, P is (α, δ) -Poisson gr-simple if 0 and P are its only graded Poisson ideals I with $\alpha(I) + \delta(I) \subseteq I$.

In a commutative graded algebra, $P_\rho P_\tau \neq 0$ forces $\rho\tau = \tau\rho$, and since $\{P_\rho, P_\tau\} = \{P_\tau, P_\rho\} \subseteq P_{\rho\tau} \cap P_{\tau\rho}$, the same holds for nonzero brackets. In particular, if P is a domain, then $\text{Supp}(P)$ generates an abelian subgroup. If P is not a domain, $\text{Supp}(P)$ need not generate an abelian subgroup: $k[x, y]/(xy)$ is graded by the free group on a, b with $\deg x = a$ and $\deg y = b$. However, if $P[t]$ is graded with P_τ in degree τ and t in degree γ , then for $0 \neq a \in P_\tau$ we have $0 \neq ta \in P[t]_{\gamma\tau} \cap P[t]_{\tau\gamma}$, so γ centralizes $\text{Supp}(P)$ for any Γ . So the conjugation in (2) has no Poisson counterpart. Throughout this section, except in Remark 4.7, Γ is abelian.

Let α, δ be derivations of P . By Oh's theorem [26, Theorem 1.1], in the form and sign convention of [23, Theorem 2.1], there is a Poisson bracket on the polynomial algebra $P[t]$ that extends the bracket of P and satisfies

$$\{t, a\} = \alpha(a)t + \delta(a) \quad (a \in P)$$

if and only if α is a Poisson derivation and δ is a Poisson α -derivation. Such a bracket is unique by the Leibniz rule. We write $B = P[t; \alpha, \delta]_p$. A pair (α, δ) is γ -compatible if $\alpha(P_\tau) \subseteq P_\tau$ and $\delta(P_\tau) \subseteq P_{\gamma\tau}$ for all τ ; since Γ is abelian, these are the degree conditions in Definition 1.1 with α in place of σ .

Proposition 4.1. *Let $\gamma \in \Gamma$. The Poisson algebra B admits a Γ -grading extending that of P with $t \in B_\gamma$ if and only if (α, δ) is γ -compatible. In that case the grading is unique, namely $B_\beta = \bigoplus_{n \geq 0} P_{\beta\gamma^{-n}} t^n$.*

Proof. If such a grading exists and $a \in P_\tau$, then $\{t, a\} = \alpha(a)t + \delta(a) \in B_{\gamma\tau}$. Comparing the coefficients of t and 1 in each homogeneous component, as in the proof of Theorem 2.1, gives $\alpha(a) \in P_\tau$ and $\delta(a) \in P_{\gamma\tau}$. Uniqueness is proved as in Theorem 2.1. Conversely, since Γ is abelian, the formula defines a grading of the commutative algebra $P[t]$. Call a pair (x, y) of homogeneous elements *good* if $\{x, y\} \in B_{\deg x \deg y}$. Pairs of homogeneous elements of P are good because P is graded, the pairs (t, a) and (a, t) are good by compatibility and antisymmetry, and (t, t) is good. If (x, y) and (x, z) are good, then so is (x, yz) , by the Leibniz rule and because Γ is abelian; the same holds in the first entry. Every homogeneous element of B is a sum of products of homogeneous elements of P and copies of t , all of the same degree, so every pair of homogeneous elements is good. $\square$

As in the associative case, innerness produces a Poisson normal element of t -degree one.

Lemma 4.2. *Let (α, δ) be γ -compatible.*

- (1) *If $\delta = \delta_u^p$ with $u = \sum_\rho u_\rho$, then $\delta = \delta_{u_\gamma}^p$.*
- (2) *If $\delta = \delta_u^p$ with $u \in P_\gamma$, then $t - u \in B_\gamma$ is Poisson normal and $B(t - u)$ is a nonzero proper graded Poisson ideal of B .*

Proof. (1) For $a \in P_\tau$, the elements $\{u_\rho, a\}$ and $u_\rho\alpha(a)$ lie in $P_{\rho\tau}$, while $\delta(a) \in P_{\gamma\tau}$. Comparing components of degree $\gamma\tau$ gives the claim.

(2) For $a \in P$, $\{t - u, a\} = \alpha(a)t + \delta(a) - \{u, a\} = \alpha(a)(t - u)$. Moreover $\delta(u) = \{u, u\} - u\alpha(u) = -u\alpha(u)$, so $\{t, t - u\} = -\{t, u\} = -\alpha(u)t - \delta(u) = -\alpha(u)(t - u)$. Since P and t generate B , the Leibniz rule gives $\{B, t - u\} \subseteq (t - u)B$. Hence $B(t - u)$ is a Poisson ideal. It is graded because $t - u$ is homogeneous, and proper because its nonzero elements have positive t -degree. $\square$

Theorem 4.3. *Let (α, δ) be γ -compatible and $B = P[t; \alpha, \delta]_p$, graded as in Proposition 4.1.*

- (1) *If B is Poisson gr-simple, then P is (α, δ) -Poisson gr-simple and δ is not α -inner.*
- (2) *Suppose that P is Poisson gr-simple. Then B is Poisson gr-simple if and only if B contains no homogeneous Poisson normal element $t^n + \sum_{i < n} b_i t^i$ with $n \geq 1$.*

- (3) Suppose that P is Poisson gr-simple and $\text{char } k = 0$. Then B is Poisson gr-simple if and only if δ is not α -inner.

Proof. For $a \in P$ and $n \geq 0$,

$$\{a, t^n\} = -nt^{n-1}(\alpha(a)t + \delta(a)). \quad (5)$$

(1) Let I be a graded Poisson ideal of P with $\alpha(I) + \delta(I) \subseteq I$. Then $IB = \bigoplus_n It^n$ is a graded ideal of B with $IB \cap P = I$. For $a \in P$ and $i \in I$, (5) gives $\{a, it^n\} = \{a, i\}t^n + i\{a, t^n\} \in IB$, and $\{t, it^n\} = (\alpha(i)t + \delta(i))t^n \in IB$. By the Leibniz rule IB is a Poisson ideal, so $I \in \{0, P\}$. If δ is α -inner, Lemma 4.2 gives a nonzero proper graded Poisson ideal.

(2) If such an element f exists, then Bf is a nonzero proper graded Poisson ideal. Conversely, let $J \neq 0$ be a graded Poisson ideal of B , let n be the least t -degree of a nonzero element of J , and let $L \subseteq P$ be the set of t^n -coefficients of the elements of J of t -degree at most n . As in Lemma 3.4, L is a graded ideal of P . If $g = ct^n + \dots \in J$ has t -degree at most n and $a \in P$, then by (5) the t^n -coefficient of $\{a, g\} \in J$ is $\{a, c\} - n\alpha(a)$; since $n\alpha(a) \in L$, also $\{a, c\} \in L$. So L is a nonzero graded Poisson ideal, and $L = P$. If $n = 0$, then $1 \in J$. Let $n \geq 1$. The component of degree γ^n of an element of J with t^n -coefficient 1 has the form $f = t^n + \sum_{i < n} b_i t^i$ and lies in J . For $a \in P$, the element $\{a, f\} + n\alpha(a)f \in J$ has t -degree less than n , so it vanishes. The t^n -coefficient of $\{t, f\} = \sum_i (\alpha(b_i)t + \delta(b_i))t^i$ is $\alpha(b_{n-1})$, and its t^{n+1} -coefficient is $\alpha(1) = 0$; so $\{t, f\} - \alpha(b_{n-1})f \in J$ vanishes as well. By the Leibniz rule f is Poisson normal. Hence, if no such element exists, $J = B$.

(3) Necessity is part (1). Conversely, suppose that B is not Poisson gr-simple. By (2) there is a homogeneous Poisson normal $f = t^n + bt^{n-1} + \sum_{i < n-1} b_i t^i$ with $n \geq 1$. For $a \in P$ write $\{a, f\} = hf$. Comparing t -degrees gives $h \in P$, and the t^n -coefficients give $h = -n\alpha(a)$. By (5), the t^{n-1} -coefficient of $\{a, f\}$ is $-n\delta(a) + \{a, b\} - (n-1)b\alpha(a)$, while that of $-n\alpha(a)f$ is $-nb\alpha(a)$. Hence $n\delta(a) = \{a, b\} + b\alpha(a)$, that is, $\delta = \delta_u^p$ with $u = -n^{-1}b$. $\square$

For a commutative noetherian $\mathbb{C}$ -algebra P with zero bracket and $\alpha = 0$, Jordan [19] shows that the prime spectrum of $P[t; \delta]$ is homeomorphic to the Poisson prime spectrum of $P[t; \delta]_P$.

4.1. Semiclassical limits. Let S be a k -algebra and $h \in S$ a central non-zero-divisor such that S/hS is commutative. Then S/hS is a Poisson algebra with bracket $\{\bar{a}, \bar{b}\} = \overline{h^{-1}(ab - ba)}$, called the *semiclassical limit* of S . Here h plays the role of Planck's constant [12, Sections 1.7 and 2.1]. For Ore extensions in which $\sigma - \text{id}$ and δ take values in hR , the semiclassical limit is a Poisson polynomial ring [23, Proposition 4.1], [7, Theorem 2.4].

Lemma 4.4. Let $w \in S$, and let ϑ be an automorphism of S such that $sw = w\vartheta(s)$ and $\vartheta(s) - s \in hS$ for all $s \in S$. Then $\bar{w}$ is Poisson normal in S/hS .

Proof. For $s \in S$, $sw - ws = w(\vartheta(s) - s)$, so $\{\bar{s}, \bar{w}\} = \bar{w} \overline{h^{-1}(\vartheta(s) - s)} \in \bar{w}(S/hS)$. $\square$

Let $S = R[z; \sigma, \delta]$ be a k -algebra, and let $h \in R$ be central in S and a non-zero-divisor, with R/hR commutative, $(\sigma - \text{id})(R) \subseteq hR$ and $\delta(R) \subseteq hR$. Put $P = R/hR$ and

$$\alpha_1(\bar{r}) = \overline{h^{-1}(\sigma(r) - r)}, \quad \delta_1(\bar{r}) = \overline{h^{-1}\delta(r)}.$$

Lemma 4.5. (1) α_1 and δ_1 are well defined, S/hS is the polynomial ring $P[\bar{z}]$, and $\{\bar{z}, \bar{r}\} = \alpha_1(\bar{r})\bar{z} + \delta_1(\bar{r})$. Hence $S/hS = P[\bar{z}; \alpha_1, \delta_1]_P$.

(2) For $u \in R$, we have $\delta_u(R) \subseteq hR$ and $\overline{h^{-1}\delta_u(r)} = \delta_u^p(\bar{r})$, where δ_u^p is computed with α_1 . In particular, if $\delta = \delta_u$, then $\delta_1 = \delta_u^p$ is α_1 -inner.

Proof. (1) Since h is central, $zh = hz$, that is, $\sigma(h)z + \delta(h) = hz$; hence $\sigma(h) = h$ and $\delta(h) = 0$. For $r' \in R$ this gives $\sigma(hr') - hr' = h(\sigma(r') - r') \in h^2R$ and $\delta(hr') = h\delta(r') \in h^2R$, so both formulas are independent of the representative; division by h is unique because h is a non-zero-divisor. Since h is central and lies in R , $hS = \bigoplus_n hRz^n$, so $S/hS = \bigoplus_n Pz^n$. This ring is commutative, because $\bar{z}\bar{r} - \bar{r}\bar{z} = \overline{(\sigma(r) - r)z + \delta(r)} = 0$. Its semiclassical bracket extends that of P and satisfies

$$\{\bar{z}, \bar{r}\} = \overline{h^{-1}(zr - rz)} = \overline{h^{-1}(\sigma(r) - r)}\bar{z} + \overline{h^{-1}\delta(r)} = \alpha_1(\bar{r})\bar{z} + \delta_1(\bar{r}).$$

By [23, Theorem 2.1], α_1 is a Poisson derivation, δ_1 is a Poisson α_1 -derivation, and $S/hS = P[\bar{z}; \alpha_1, \delta_1]_P$; compare [23, Proposition 4.1] and [7, Theorem 2.4].

(2) We have $\delta_u(r) = (ur - ru) - (\sigma(r) - r)u$, the first term lies in hR because R/hR is commutative, and the second because $(\sigma - \text{id})(R) \subseteq hR$. Hence

$$\overline{h^{-1}\delta_u(r)} = \{\bar{u}, \bar{r}\} - \alpha_1(\bar{r})\bar{u} = \delta_u^p(\bar{r}).$$

If $\delta = \delta_u$, then $\delta_1(\bar{r}) = \overline{h^{-1}\delta_u(r)} = \delta_u^p(\bar{r})$. $\square$

Example 4.6. Let $F = \mathbb{Q}[q^{\pm 1}]$, $h = q - 1$ and $R = F[x]$, graded by $\deg x = 1$ and $\deg q = 0$. Let σ be the F -algebra automorphism with $\sigma(x) = qx$, and let δ be the F -linear σ -derivation given by $\delta(f) = (f(qx) - f(x))/x$, so that $\delta(x^m) = (q^m - 1)x^{m-1}$ for $m \geq 1$, and put $S = R[z; \sigma, \delta]$, graded by $\deg z = -1$. Thus

$$zx = qxz + (q - 1),$$

and h is central in S because $\sigma(q) = q$ and $\delta(q) = 0$. Specializing q to 2 gives the quantized Weyl algebra of Example 2.5. In S , the element $w = 1 + xz$ satisfies $wx = qwx$ and $zw = qwz$. Hence $sw = w\vartheta(s)$ for the F -algebra automorphism ϑ of S with $\vartheta(x) = q^{-1}x$ and $\vartheta(z) = qz$, which preserves the defining relation. Since $\vartheta(x) - x$ and $\vartheta(z) - z$ lie in hS and $\vartheta(ab) - ab = (\vartheta(a) - a)\vartheta(b) + a(\vartheta(b) - b)$, we get $\vartheta(s) - s \in hS$ for every $s \in S$.

The semiclassical limit S/hS is $\mathbb{Q}[x, z]$ with $\{z, x\} = xz + 1$, that is, $B = \mathbb{Q}[x][z; \alpha_1, \delta_1]_p$ with $\alpha_1 = x \frac{d}{dx}$ and $\delta_1 = \frac{d}{dx}$, and (α_1, δ_1) is (-1) -compatible. By Lemma 4.4, $\bar{w} = 1 + xz \in B_0$ is Poisson normal. It is not a unit, so $B\bar{w}$ is a nonzero proper graded Poisson ideal and B is not Poisson gr-simple. On the other hand, $\mathbb{Q}[x]$ with the zero bracket is (α_1, δ_1) -Poisson gr-simple, since no ideal (x^m) with $m \geq 1$ is δ_1 -stable, and δ_1 is not α_1 -inner, because $\mathbb{Q}[x]_{-1} = 0$ and $\delta_1 \neq 0$ (Lemma 4.2(1)). This is the Poisson counterpart of Proposition 3.2. As in Remark 3.3, $1 + xz = x(z - u)$ with $u = -x^{-1}$, and δ_1 is α_1 -inner over $\mathbb{Q}[x^{\pm 1}]$. The diagram below summarizes the two specializations of the graded family in Example 4.6. After extending the coefficient rings to $F[x^{\pm 1}]$ and $\mathbb{Q}[x^{\pm 1}]$, respectively, the derivations in the family and in its specialization at $q = 2$ satisfy $\delta = \delta_u$ with $u = -x^{-1}$. On the Poisson side, $\delta_1 = \delta_u^p$ over $\mathbb{Q}[x^{\pm 1}]$ (Lemma 4.5(2), applied to the extended coefficient ring). The factorization $w = x(z - u)$ explains the normal element before specialization and its Poisson analogue. Both arrows are quotient maps of associative algebras. The right-hand target is equipped with the semiclassical bracket $\{\bar{a}, \bar{b}\} = \overline{(ab - ba)/(q - 1)}$. The image of w generates a nonzero proper graded ideal in A and a nonzero proper graded Poisson ideal in B .

$$\begin{array}{ccc}
 & \begin{array}{c} S = F[x][z; \sigma, \delta] \\ zx = qxz + (q - 1) \\ w = 1 + xz \text{ normal} \end{array} & \\
 \swarrow \scriptstyle q \mapsto 2, z \mapsto t & & \searrow \scriptstyle \text{mod } (q-1)S \\
 \begin{array}{c} A \cong S/(q-2)S \\ tx = 2xt + 1 \\ 1 + xt \text{ normal} \end{array} & & \begin{array}{c} B = S/(q-1)S \cong \mathbb{Q}[x, z] \\ \{z, x\} = xz + 1 \\ 1 + xz \text{ Poisson normal} \end{array}
 \end{array}$$

Remark 4.7. The conjugation phenomenon of Theorem A does not survive in a semiclassical limit. Let $S = R[z; \sigma, \delta]$ be graded as in Theorem 2.1 with $\deg z = \gamma$, let $h \in R_e$ be central in S and a non-zero-divisor, and suppose that $(\sigma - \text{id})(R) \subseteq hR$. If $\gamma\tau\gamma^{-1} \neq \tau$ and $r \in R_\tau$, then $\sigma(r) \in R_{\gamma\tau\gamma^{-1}}$, so $\sigma(r) - r \in hR$ has component $-r$ in degree τ . Since h is homogeneous, hR is graded, and $r \in hR$. Hence $R_\tau \subseteq hR$, and the grading of R/hR is supported on the centralizer of γ .

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